CALCULATION OF PERTURBATION

by E. Fues

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Chapter 4

CALCULATION OF PERTURBATION

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with 4 Figures

I. INTRODUCTION

1. The significance of the theory of perturbation for physics. Strictly speaking, the methods of analytic mechanics are only adequate to overcome the most simple motion problems of point systems. It is, in fact, possible to prove the famous Keplerian Laws for the two-body problem with relatively simple means,[1] but the three-body problem already escapes exact mathematical integration.[2] For a long time astronomers have tried to tear down these apparently insurmountable limits of our analysis; however, it has been shown that the difficulty does not lie in the imperfection of mathematical methods but in the mechanism of the motion itself. Poincaré has proven that the three-body problem does not admit a large enough number of significant, integrals, which are necessary to represent for any given time the coordinates as multiple periodic functions of time. Thus, it is no

1) Cf. Ch. 7, Fig. 25 of this Handbook Volume.
2) Cf. Ch. 7, Fig. 26 of this Volume.

wonder that the over a century old experiments in this direction were futile.

Under the stress of this impossibility approximation methods were sought at an early stage. The minuteness of the forces acting on a planet from its neighbor planets with respect to the sun's force of attraction permits one to expand the motion equations in power series according to the small relationship of the masses, and from this a similar expansion of the integrals can be derived. Moreover, it has been shown that this approach is not limited to the case in which the nature of the problem permits strictly unambiguous integrals, but that it also continues to exist formally when the system observed is of such a kind as say, the three-body problem. In any event, these formal solutions possess no absolute convergence--hence the impossibility of exact integration--but they are of the greatest significance in practical celestial mechanics because of their semi-convergence. The branch of mechanics included

under the name of perturbation theory[1] is concerned with nothing more than erecting a formal integration process for given mechanical problems which can be viewed as "perturbations" of a known integrable mechanism. The development of this method is particularly associated with the names Lagrange and Delaunay in an earlier period and in much greater completeness in later times with the astronomers Gylden, Lindstedt and Bohlin as well as the mathematician Poincaré.

Earlier physics had no cause to be interested in these methods of calculation until the construction of the Bohr model of the atom suddenly established a close relationship between atomic theory and cosmic astronomy.[2] Bohr himself was the first

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1) Some of the detailed texts on the subject are: H. Poincaré, Lecons de mécanique céleste. Vol. 3. Paris: Gauthier Villars 1905; H. Poincaré, Les méthodes nouvelles de la mécanique céleste. Vol 3. Paris: Gauthier Villars 1892; C. L. Charlier, Die Mechanik des himmels, Vol. 2. Leipzig: Veit & Co. 1907; E. T. Whittaker, Analytische Dynamik der Punkte und starren Körper. German by Mittelsten Scheid. Berlin: Julius Springer 1924. The following texts and theses already discuss the subject in view of the application to the (Bohr) atomic theory: J. M. Burgers, Het Atoommodel van Rutherford-Bohr. Dissert. Leiden. Haarlen: Erven Loosjes 1918; A. Sommerfield, Atombau und Spektrallinien, fourth edition, Braunschweig: Vieweg 1923; the most detailed: M. Born, Vorlesungen über Atomdynamik, Sammlung Struktur der Materie. Berlin: Julius Springer 1925. The author is obligated to countless stimulating ideas and insightful suggestions from an unpublished article by W. Pauli, Jr., which was originally thought of as an introduction to the article on quantum theory in this Handbook, Vo. XXIII.

2) Cf. the article on quantum theory by W. Pauli, Jr., in this Handbook, Vol. XXIII.

to point out the resources completed by the astronomers which were available for the purposes of atomic research. The influence of an external electric field and that of the relativistic forces of gravity on the Kepler paths in the hydrogen atom can be calculated with the method of the secular perturbations. However, with the many-body problem of the Bohr atoms and molecules the situation is much less favorable than in celestial mechanics. The expansion parameter, the relationship of the electron charge to the nucleus charge, is by far not as small as in celestial mechanics, which influences the convergence of the series unfavorably. Secondly, the time periods one is interested in--measured on the characteristic periods of the system--are enormously larger than in astronomy. In spite of this, a series of atom problems are dealt with according to perturbation theory. On one hand Epstein and on the other Born and his colleagues (Brody, Pauli, Heisenberg, Nordheim) have worked on the transposition of astronomical methods to atomic physics. The direct yields were, first of all, not as large as one might have hoped for--or, moreover, they lay in another direction. Aside from general information on decay and development of motion, on the "effects" in the spectra, on the phase relationships and the general character of motion in the molecular bond, one gained no understanding of the helium spectrum or the hydrogen molecule ion. One did, however, come to the always certain conclusion that

classical mechanics, even in connection with "quantum conditions," is not capable of leading to a precise understanding of complicated atoms. Even if there was no success in the main concern, a mathematical resource was created, which can be applied to growing circle of tasks and which certainly will some day prove out in regions of physics which are still a long way away.

Meanwhile, atomic physics has acquired a new formulation. According to Heisenberg's process, Born, Jordan, Dirac and others have developed a theory of spectral frequencies and intensities,[1] which is indeed very similar externally to earlier mechanics, but which follows such unusual rules of calculation that it can only be understood with the help of altered basic definitions. Schrödinger could later prove that this formalism is associated with the problem of the natural oscillation frequency of a continuum, thus, something quite different than could be suspected from the initial equations.[2] However, there is an inner connection between the "wave mechanics" of the atomic world and its macroscopic law of limits, point mechanics, which allows the understanding of the close formal relationship of both. Wave mechanics also has its theory of perturbation, but cannot be discussed in the present chapter.

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1) Cf. M. Born, Probleme der Atomdynamik. Berlin: Julius Springer 1926.
2) Cf. E. Schrödinger, Abhandlungen zur Wellenmechanik. J. A. Barth. Leipzig 1927. Cf. the article "Optik und Mecnanik" by A. Lande in this Handbook, Vol. XX.

The observations in this chapter make extensive use of the treatment of mechanical questions presented in Chapter 3 by Hamilton and Jacobi. They therefore are linked to the canonical form of the motion laws of a mechanical system of f degrees of freedom (cf. Ch. 3, No. 12).

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k}. \qquad (k = 1, 2, \ldots f) \tag{1}$$

The integration process always consists of an integration of the Hamilton partial differential equation (cf. Ch. 3, No. 12), and the theory of the canonical transformation of a problem will be used.

Before one can go into the actual subject of this chapter, perturbation calculation, it is necessary to present in advance several comments on the forms of motion and on the distinct role which the periodic and limited periodic movements play in higher mechanics.

II. MULTIPLE PERIODIC MOVEMENTS

2. The significance of unique integrals. Consider the condition of the mechanical system at a given starting time t_0, characterized by the position of an image point in the $2f$ dimensional phase volume of p_k, q_k, the following system transformation by the migration of an image point on the phase orbit calculated from the equations (1), No. 1. In addition,

for present purposes several simplifying presuppositions will be made. Let this be a closed system for which the Hamilton function H is independent of time, thus

$$H(p, q) = W \qquad (1)$$

is an integral, namely the integral integral in general. The phase orbit then runs on the (2f -1) dimensional hypersurface given by Equation (1), which we will shortly like to call the energy surface, not bothered by the fact that W does not signify the system energy in every case. Moreover, we will suppose that the energy surface, and with it the phase path run completely in the finite. This excludes, on the one hand, all of the movements which run to the infinite region of the illustrative volume (e.g., the hyperbolic movements of the planets) and, on the other hand, also includes an assumption on the selection of coordinates: All angle--like coordinates which can grow in an unlimited fashion in spite of the limited position of the system are not to be applied for the moment, but only those which are uniquely related to the position of the system.

If one constructs the phase paths at all the possible initial conditions of the system, they fill the phase volume closely in such a way that, excepting certain singular points, only one phase path goes through each phase point. In its further progress each phase path remains constantly in the energy surface

belonging to it; it shares this fate with all phase paths which one can project through initial points with the same system energy. For example, beginning with the initial point first selected one can still continue in (2f -2) dimensions to the neighboring points transversally to the phase path without leaving the integral surface H = W. Moreover, its surface element is directly fulfilled by the selected number of points and by the path elements passing through it. Using the dimension perpendicular to the energy surface one can combine another (2f -2)-fold number of points and their path elements into the element of a hypersurface equally well, though. If one always continues this in such a way that it always follows the phase paths lying in it, this construction is derived from the new integral surface.

$$F(p, q) = \alpha_i$$

One can easily imagine that all together there exist (2f -1) independent families of such integral surfaces. The phase paths are the family of their intersections. The equations of the integral surfaces together with an equation for the chronological course of the motion are the 2f possible integrals of the mechanical problem.

Let us select one of the energy surfaces and follow a phase

path on it further and further. One can expect that,[1] in general, that it fills the entire energy surface by lining up more and more orbital loops. (The expression "surface" destroys one's ability to visualize this to a great degree here. With two degrees of freedom one is already dealing with a three-dimensional energy volume.) It then proceeds quasi-ergodically, to use a term from statistical mechanics, that is, in the course of time it approaches every point on the energy surface or on a related surface of it.[2] Every integral surface $F_k = \alpha_k$, itself constantly overlapping intersected by $H = W$ must finally fill a $2f$ -dimensional region densely since it follows the phase paths. Thus, the various surfaces of a family penetrate each other; the value of α_k in a definite phase path element becomes infinitely ambiguous. Aside from $H = W$ there are no unique integrals.

This possiblility is opposed to the other case. If a phase path does not densly fill its energy surface or a $(2f - 1)$ dimensional region in it, but only a $(2f - n)$ dimensional section of it, it can therefore be thought of as arising from an intersection of n, integral surfaces F_1 to F_n not overlapping infinitely many times

1) This is mainly based on observations of statistical mechanics, especially on the Liouvillean Theorem proven in Ch.3, No. 5. E. Fermi has attempted a proof, Phys. 25, Vol. 24, p. 261, 1923.

2) E. Artin gives an example of a quasi-ergodic system, Abhandl. a.d. math. Sem. d. Hamburger Univ. (Dissertation at the mathematical seminar of the Univ. of Hamburg) Vol. 3, p. 170, 1924.

with the energy surface. Neighboring surfaces to such families do not intersect each other unless at certain singular positions; each path element has only one parameter value α_k. There are n other unique integrals independent of H = W, not explicitly containing time.

The significance of infinitely ambiguous integrals is much different than that of the unique integrals. These latter limit movement to a much greater degree than the former. The existence of the former (there are always 2f integrals of motion, irregardless of how the system was created) actually says nothing more than that the motion is uniquely determined. The latter only exist in particularly simple mechanical systems or for special cases of more general motions. How many unique integrals are possible is, thus a question of the greatest importance in the investigation of a mechanical system. In certain cases the non-existence of unique integrals can be proven (cf. No. 16).

3. The special position of the multiple periodic movements. The motion is especially simple when at least f unique integrals exist. No matter how one makes a derivation from the image of their integral surfaces in the pq volume, in this case only one or a finite number of values of the impulse vector can belong to a point in the q volume during the same system motion, just as, when a pendulum oscillates undisturbed, at every point of its stroke only two speeds are possible. Further simplifications

of the motion seem to be necessarily bound as a function of the position to this finite, ambiguous definiteness of the impulse vector. Mechanical systems of this kind always are multiple periodic if one ignores special cases and all motions proceeding infinitely. The recently abandoned atomic mechanics and astronomy deal almost exclusively with them, and their mathematical accessibility is so much greater than that of the complicated types of motion that the attempt to master them also with "perturbation calculation" always terminates in an approximation by multiple periodic movements. Therefore, they alone will be discussed in the following.

The theorem that f unique integrals produce multiple periodic motion has been strictly proven for two degrees of freedom by Kneser.[1] His arguments make it probable, however, for additional degrees of freedom. The following sections will show that the existence of further unique integrals only reduce the degree of periodicity. Nevertheless, in a closed system with fewer than f unique integrals no multiple periodic motion seems to be possible, not even with more than f periods, and, moreover, the reversal of the above theorem seems to hold: Multiple periodic motion is bound to the existence of at least f unique integrals. Compare Ehrenfest's[2] and Wataghin's[3] notes, which don't give any proof for this. (The

1) H. Kneser, Math. Ann. Vol. 84, p. 277, 1921.

2) P. Ehrenfest, ZS f. Phys. Vol. 19, p. 242, 1923.

3) G. Wataghin, Ann. d. Phys. Vol. 76, p. 41, 1925.

theorem does not hold for non-closed systems. There a more than f-fold periodic motion is possible, as the example of the constrained vibrations of an oscillator already shows.)

4. Variable of angle and effect. In Section 3 it was stated that celestial mechanics attempts to describe the motions of its systems (with certain exceptions) as multiple periodic, be it rigorously or by approximation. The basis for this description of a certain type of motion was recognized in its special mathematical position which permits an especially complete integration. When one reads over the texts on celestial mechanics, one discovers a tiresome abundance of transformations which seek to adapt as exactly as possible to the selection of variables for each special problem. The physicists who transfered these methods to atomic physics immediately completed a certain sorting operation in the interests of the quantum theory and selected the most important transformations. This led to a very coherent form of calculation, which is possible for every multiple periodic system: calculation in angle and effect variables. With the help of this calculation the calculation of perturbations in particular acquired a highly comprehendable construction. The definition and introductions to this standardizing variable, as one says, will be described step by step in the following sections.

5. Periodic motion with one degree of freedom. In systems with one degree of feedom the equation $H(p,q) = W$ is also the equation for the phase path in the pq phase plain. A family of curves corresponds to various values of W; their individual curves do not intersect each other on account of the uniqueness of W. We then assume that q is a coordinate which is dependent on the system in a unique manner, i.e., the cartesian coordinate of a mass point or the position on a non-closed curve etc., but is not the angle of rotation which can assume various values for the same position of the system. If we limit consideration to motions which completely proceed into the finite region and are uniform in the physical sense, the curves $H = W$ are necessarily closed within themselves (cf. Fig. 1)

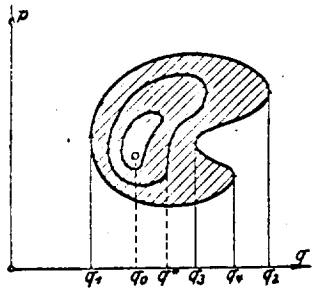

Fig. 1 Phase paths of a periodic motion with libration.

The motion proceeds between fixed limits (q_1 q_2 q_3 q_4 q_1); one calls this libration. With the variable W the phase path on a small curve contracts around a fixed point q_o, the center of libration. The motion itself degenerates to small oscillations around a stable position of equilibrium in q_o. The limits of libration are given

by $q = \partial H/\partial p = 0$, they always occur as couples when W changes (e.q., q_3, q_4) and contract at the instant of their creation. The point where they contract corresponds to a labile position of equilibrium (q^*).

However, if the degree of freedom q is angular, so that $q + x$ actually describes the same system position as the value q, the unique definiteness of the integral value W resulting from the state of motion requires either that the curve is again closed or that p is represented with the period x as the periodic function of q (cf. Fig. 2). In the second case the angular coordinate q increases in an unlimited fashion; however, the position of the system is repeated during this case from time to time. The motion type is that of rotation. Also, libration

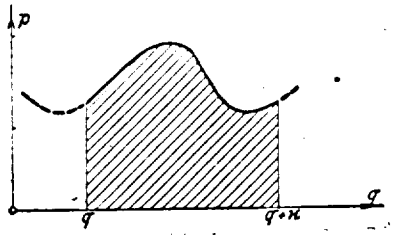

Fig. 2 Phase path of a periodic motion with rotation.

motions with W varied often are transformed into rotation motions (cf. the following example). As borderline case between both limitation can appear, i.e., a motion which only approaches a point of inflection still present in an infinitely long time.

The most well known example for the three types of

motion is the pendulum motion[1] with the energy equation.

$$H \equiv \frac{1}{2A} p_\varphi^2 - D\cos\varphi = W$$

(A is the moment of inertia, D the product of pendulum weight and distance between fulcrum and center of gravity, φ the swing). From it can be calculated

$$p_\tau = \sqrt{2A}\sqrt{W + D\cos\varphi},$$

represented in Fig. 3. When W = D there is inertia in the libration center, for - D<W<+ D various extended librations arise. W = D corresponds to the known infinitely slow approach of the pendulum to the uppermost point, i.e., the limitation, and W>D corresponds to the rotary motion of the pendulum, the rotation.

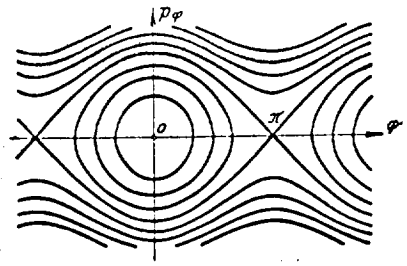

Fig. 3 Phase paths of the pendulum motion.

So much on the spatial character of the motion! The periodic phase path returning back on itself or q allows one to expect a path periodic in time without calculation; and calculation confirms

1) See Ch. 7, No. 12 in this Volume of the Handbook.

this. The case of one degree of freedom is, thus, a revealing example for the general theorem suspected in No. 3 that f unique integrals of the motion result in the chronologically periodical or multiple periodical course of this phase path.

We will now turn to the integration of the motion and follow the path suggested in Ch. 3, Sect. 12; it easily leads to the form of calculation discussed in Section 4. We will thus look for the integral S (q,'α) of the Hamiltonian differential equation.

$$H\left(\frac{\partial S}{\partial q}, q\right) = W$$

and use the transformation equations as the generatrices [cf. Ch. 3, Sect. 3 (6)].

$$p = \frac{\partial S}{\partial q}, \qquad Q = \frac{\partial S}{\partial \alpha}.$$

The operation nearest at hand would be to use the quantity W as the chronologically constant, new impulse α, but another choice is better for reasons which will be clear later, namely that of the integral

$$J = \oint p\,dq. \tag{1}$$

The symbol $\oint$ means that an integration is to be made along the entire phase path until the return of the system to its initial slate; thus with the libration motion once over the closed phase path, with rotation over a period of the coordinate q.

The newly selected impulse J is called the variable of effect

because it indicates the increase in the characteristic function (more generaly the function S) during a complete system cycle. It is

$$\oint \frac{\partial S}{\partial q}\, dq = J.$$

Its chronologically invariable value is equal to the content of the surface elements shaded in Figs. 1 and 2, naturally independent of W, so that conversely W = W(J).

The canonically conjugated coordinate of position.

$$w = \frac{\partial S(q, J)}{\partial J} \tag{2}$$

is called the angle variable. It has the following characteristics: on one hand it increases linearly with time since the transformed Hamiltonian function is

$$K(J, w) = W(J),$$

so that the "average motion" of W, namely $\dot{w} = \partial W/\partial J = Y$ is a constant and thus

$$w = \nu t + \delta \tag{3}$$

While the increase of W amount to $v = \partial W/\partial J$ per unit of time, it amounts to

$$\oint \frac{dw}{dq}\, dq = \oint \frac{\partial^2 S}{\partial J \partial q}\, dq = \frac{\partial}{\partial J} \oint \frac{\partial S}{\partial q}\, dq = 1.$$

during a complete system rotation.

In other words: Each time that the system has completed a complete motion and returned to the initial state, w increases by the same amount of one. From this, it follows that the state of the system

is periodic in w with period 1, so that one can write[1]

a) in the case of libration; $q = q(\tilde{w}) = \sum_{-\infty}^{+\infty} a_\tau e^{2\pi i \tau w},$

b) for a rotating q: $q = \varkappa w + (\tilde{w}) = \varkappa w + \sum_{-\infty}^{+\infty} a_\tau e^{2\pi i \tau w}.$ (4)

The coefficients of the Fourier series are dependent on J and are defined in a familiar fashion as

a) $a_\tau = \int_w^{w+1} q(\tilde{w}) e^{-2\pi i \tau w} dw,$

b) $a_\tau = \int_w^{w+1} (q - \varkappa w) e^{-2\pi i \tau w} dw.$

One can easily imagine that S can be represented in form

$$S = Jw + (\tilde{w}). \tag{5}$$

Finally, the chronological periodicity of the motion can be read from (3) and (4); it is

a) in the case of libration: q (6)

b) in the case of rotation: $q - \varkappa(\nu t + \delta) = \sum_{-\infty}^{+\infty} a_\tau e^{2\pi i(\nu t + \delta)},$

whereby $\nu = \partial W / \partial J$ signifies the frequency of the motion.

For two motions of the system which are described by various, but neighboring values J, $J + \Delta J$ the here self-evident theorem

$$\Delta W = \nu \Delta J \tag{7}$$

holds true. In quantum theory it is the point where Bohr's principle of correspondence is tied in.[2]

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1) In the following the abbreviation f $(\tilde{w})$ always means: periodic function of w. If it says nothing else, this does presuppose that the period is equal to one.

2) Cf. the article "Quantentheorie" by W. Pauli, Jr. already mentioned at the beginning. This Handbook, Vol. XXIII.

An example will bring out the significance of the new variables: The equation of energy of the linear harmonic oscillator is

$$H = \frac{p^2}{2m} + 2\pi^2 \nu^2 m q^2 = W$$

(m the mass, v the frequency, q the swing).

One obtains

$$J = \oint p\,dq = \frac{W}{\nu} \quad \text{und} \quad S = \sqrt{2m}\int \sqrt{\nu J - 2\pi^2 \nu^2 m q^2}\,dq .$$

From this follows

$$w = \frac{\partial S}{\partial J} = \frac{1}{2\pi} \arcsin \sqrt{\frac{2\pi^2 \nu m}{J}}\, q = \nu t + \delta$$

and

$$\left.\begin{aligned} q &= \sqrt{\frac{J}{2\pi^2 \nu m}} \sin 2\pi w, \\ p &= \sqrt{2\nu m J} \cos 2\pi w. \end{aligned}\right\} \qquad (8)$$

The introduction of w, thus, corresponds to the known geometric construction of the sine oscillation as a projection of a uniform circular cycle, i.e. a rotation. The unit of the angle of rotation is selected so that the period in it equals one.

6. Separable multiple periodic systems.[1] If equation for the constancy of the Hamiltonian function

$$H(p_1, \ldots p_f, q_1, \ldots q_f) = W$$

breaks down into f unique first integrals

$$H_j(p_j, q_j) = A_j; \qquad (j = 1, 2, \ldots f)$$

as assumed in Ch. 3, Sect. 13, the phase path of the system can

1) Cf. the first investigations of same by P. Stäckel, Dissertation to gain university lecturer status, Halle 1891; also C. L. Charlier, Die Mechanik des Himmels, Vol. I, Sect. 2; K. Schwarzschild, Berl. Ber. 1916, p. 548; P. S. Epstein, Ann d. Phys. Vol. 51, p. 168. 1916.

Hamiltonian function

$$, w_k, J_k \ldots) = W(J_1, \ldots J_f),$$

from which,

$$\dot{w}_k = \frac{\partial W}{\partial J_k} = \nu_k$$

and

$$\dot{w}_k = \nu_k t + \delta_k \tag{3}$$

again follow. In comparisons, the growth of W_k is

$$\oint \frac{\partial w_k}{\partial q_j} dq_j = \oint \frac{\partial^2 S}{\partial J_k \partial q_j} dq_j = \frac{\partial}{\partial J_k} \oint \frac{\partial S}{\partial q_j} dq_j = \begin{cases} 1 \text{ für } k = j, \\ 0 \text{ für } k \neq j. \end{cases} \tag{4}$$

during a complete cycle of q_j under a forced arresting of all other q. The image of a q volume on the w volume presented by (2) has the following characteristics therefore: If one begins with a definite configuration of the system and allows an individual coordinate to follow its possible course to a complete return, only the respective w increases by a unit, while all other w return to their initial value. Thus, all points of a regular lattice in the w volume with the lattice constant 1 signify the same configuration and, because of the phase path characteristic mentioned initially, to return to the same value of p_k after every cycle of q_k, also the same impulses. Since the inverse functions of (2) prove to be unique, it follows from the inversion of this relationship that q_j and p_j are periodic functions of w_k, each with periods of 1. This holds true not only for the separation coordinates used up to now, but also for all coordinates uniquely related to them. In

such cases it is, thus, possible to represent the q_j, p_j as repeated Fourier series of W_k:

$$\left.\begin{aligned} q_j = (\tilde{w}_1, \dots \tilde{w}_f) &= \sum_{-\infty}^{+\infty} \cdots \sum_{-\infty}^{+\infty} a^{(j)}_{\tau_1 \dots \tau_f} e^{2\pi i (\tau_1 w_1 + \cdots + \tau_f w_f)} \\ \text{or shortened} \quad &= \sum_{-\infty}^{+\infty} a^{(j)}_{(\tau)} e^{2\pi i (\tau w)}. \end{aligned}\right\} \tag{5}$$

The coefficients of the series are determined (dependent on the J_k) to be

$$a^{(j)}_{\tau_1 \dots \tau_f} = \int_{w_1}^{w_1+1} \cdots \int_{w_f}^{w_f+1} q_j (w_1, \dots w_f) e^{-2\pi i (\tau_1 w_1 + \cdots + \tau_f w_f)} \, dw_1 \dots dw_f.$$

When q_k itself is "angular" and rotates (q_k - x_k W_k) always takes the place of q_k in these formulas. Similar to Sect. 5

$$S = \sum_k J_k w_k + (\tilde{w}_1, \dots \tilde{w}_f). \tag{6}$$

Because of the linear increase of W_k according to Equation (3) the repeated chronological periodicity of the motion

$$\left.\begin{aligned} &q_j \\ \text{or} \quad &q_j - x_j(\nu_j t + \delta_j) \end{aligned}\right\} = \sum_{-\infty}^{+\infty} \cdots \sum_{-\infty}^{+\infty} a^{(j)}_{\tau_1 \dots \tau_f} e^{2\pi i [(\tau_1 \nu_1 + \cdots + \tau_f \nu_f) t + \gamma_{\tau_1 \dots \tau_f}]} \tag{7}$$

follows from (5). The motion frequencies are given by the quantities

$$\nu_j = \frac{\partial W}{\partial J_j}; \tag{8}$$

they are characterized as average motions of the coordinates W_j.

7. Decay of the motion. It will be purposeful for what comes later to observe more closely the image of the q volume on the W volume which is expressed by Sect. 6, Eq. (2). Since the condition of the system is periodic in the W_k with a period of 1, the unit cube of the W volume exhausts all the possibilities of the positions q. The entire path region is reproduced in it, even in

a part of it, since when the W_k assumes all positions on the unit cube, the image point in the q volume travels through the path region repeatedly on account of the librations.

It is especially easy and clear to follow the motion in the W volume. Because of the chronologically linear growth of W_k according to Sect. 6, Eq. (3), the image in the W volume moves uniformly on a straight line whose slopes with respect to the axes are given by

$$dw_1 : \cdots : dw_f = \nu_1 : \cdots : \nu_f .$$

Meanwhile, it is superfluous to follow the straight line to its full extension. Since every newly entered unit cube only represents the old state of motion again, it is sufficient to cut the path in pieces by the side surfaces of the W cubes and to shift each section back into the initial cube by integral displacement parallel to the axes. Thus an image of the path is created which consists solely of straight, parallel portions.

One notes (the proof, for example, is found in the appendix to Born's book, quote in Sect. 1, footnote) that with time the path segments fill the unit cube with uniform density when no linear, integral relationship

$$(\tau\nu) = \tau_1\nu_1 + \cdots + \tau_f\nu_f = 0$$

exists between $\nu_k = \partial W / \partial J_k$. This means that the Fourier series Sect. 6 (7) are actually f-fold periodic since f independent motion frequencies exist. One can infer the motion itself from the W volume;

the path curve densely fills the f dimensional path region in time. At some time it approaches each point of the same. (Thus also, after a quasi-period an arbitrarily selected initial point on the path.) In this sense the motion is fully evolved. One notices that, if in Sect. 6 the coordinates g_j signify actual positions in the geometric sense (and are not defined by a tangential transformation), the path region is enclosed by the limit surfaces of libration q_j = constant, and that one can say in this case that the separation coordinates are uniquely defined by the surfaces of the motion itself (cf. Ch. 3, Sect. 13).

The uniformly dense filling of the unit cube in the W volume by the path curve, together with the uniform motion in it, permits a very simple calculation of mean values of time over the motion. The integral

$$\frac{1}{T}\int_{T} f(\ldots p_j, q_j \ldots)\,dt$$

is all the more exactly equal to the mean value of volume of f take over the unit value in the W volume the larger T becomes.

On the other hand, when s linear integral relationships between the frequencies v_j exist:

$$\tau_{\varrho 1}\nu_1 + \cdots + \tau_{\varrho f}\nu_f = 0. \qquad (\varrho = 1, 2, \ldots s) \tag{1}$$

Then the unit cube of the W volume will not be densely filled, but only a (f - s) dimensional region of the same. For this reason the path curve is limited to a (f - s) dimensional region in the

position volume; the motion has decayed s-fold, as is said. The Fourier expansion Sect. 6 (7) actually does not represent any f-fold periodic function of time since s frequencies of motion can be rationally expressed by the others. The state of the system is only an (f - s)-fold periodic function of time. Each purely periodic motion of a system with more than one degree of freedom belongs to this. On account of the characteristic of multiple periodic systems to assume purely periodic motions for certain frequency-response-ratios, Staude has defined them as being limited periodic.

In the case of decay one can perform and often used extraction of the angle variables. If one substitutes s of the variables w^o introduced in the first instance, say W^o_{f-s+1} to W^o_f by the following new

$$w_\varrho = \tau_{\varrho 1} w_1^0 + \cdots + \tau_{\varrho f} w_f^0, \qquad (\varrho = f-s+1, \cdots f) \tag{2}$$

a transformation which with the help of the generatrices

$$S = \sum_{\alpha=1}^{f-s} J_\alpha w_\alpha^0 + \sum_{\varrho=f-s+1}^{f} J_\varrho (\tau_{\varrho 1} w_1^0 + \cdots + \tau_{\varrho f} w_f^0) \tag{3}$$

can also be expanded canonically to the action variables, it follows that

$$\dot{w}_\varrho = \tau_{\varrho 1} \nu_1^0 + \cdots + \tau_{\varrho f} \nu_f^0 = 0,$$

thus

$$w_\varrho = \text{constant} \tag{4}$$

They are thus called the uncharacteristic angle variables. Their constancy is only another expression for the reduction of the number of independent frequencies of motion. One notes that f - s characteristic

angle variables are sufficient to describe the motion. In the latter sections we will follow the custom of many authors and describe uncharacteristic, as opposed to characteristic, angle variables by the index e (if necessary σ, τ...); the characteristic variables will be indexed with α, β, γ...

Purely periodic motions only possess one independent frequency of motion, accordingly only one characteristic angle variable. The canonically action variable coordinate is, as one can see from Sect. 6 or, even better, Sect. 10.

$$J = \oint \sum_j p_j dq_j . \tag{5}$$

8. Characteristic, random or limit decomposition. There are three typical cases of decomposition:

If one introduces the new action variables by means of the transfomation Sect. 7, Eq. (3), then W becomes a function of the same: $W(J_1, \ldots J_f)$. However, it can be derived from Sect. 7, Eq. (4) and $\dot{W}_e = \partial W / \partial J_e$ that for all uncharacteristic angle variables

$$\frac{\partial W}{\partial J_e} = 0. \tag{1}$$

There can be three reasons for this:

Either (1) holds true for all values of J^o. Then, every possible motion of the system has decayed; a f-fold periodic motion is not at all capable of this. The division given between characteristic and uncharacteristic angle variables can always be performed in the same way. In this case the equations (1) mean that W is not at all dependent on J_e. This case is described as characteristic decay.

Or, secondly, there are rational frequency-response-ratios between the v^o values for certain values of J^o. Then, only these special motions of the system are decayed, and the equations (1) signify no functional independency of quantity W from J_e, but rather an evanescence of the derivates for certain J_e values. One then speaks of random decay. If one considers whether this case can occur frequently, then it turns out that the v^o values, for which some commensurability as in Sect. 7 Eq. (1) exists, even lie densely; the same holds true, thus for the J^o values functionally connected with them. On the other hand, there are only a few individual systems J^o for which a definitely selected variable W decays at random.

A third type of decay can also occur; it is actually the nearest at hand. When a coordinate q_j stabilizes at its center of libration instead of oscillating around it, the motion decays. The decay in the q volume need not be noticeable in the W volume. It is not a frequency that vanishes, but rather, the amplitudes in the Fourier expansion of q_j as a result of special values of J_e. Note, for example, equation (8) of the example in Sect. 5. For $J = 0$, $q = 0$ since its amplitude is proportional to $\sqrt{J}$. But W is not a constant since the constantly fixed frequency V of the oscillator does not evanesce with J. Moreover, $W - vt + d$ remains a quantity which increases linearly with time. This expresses a fundamental difference in the angle variables of each librating coordinate.

With the help of angle variables the motion is described as a "uniform rotation;" however, for the rotation there is no transition to the rest state, even with vanishing frequency. In the case of decay on account of vanishing amplitude, which one characterizes as limit decay, the difference causes the image of the q volume on the W volume to lose its uniform character. Thus, one comes to the opinion that the methodical introduction of angle and action variable used so excellently in other cases, here leads to disadvantages (cf. Sect. 21 and 22).

One sees that limit decay can also be coupled with real or random decay. Then amplitudes and the frequency of the Fourier expansion vanish simultaneously.

9. Keplerian motion. In order to explain the somewhat formal introduction of the angle and action variables by an example, in the following the Keplerian motion[1] of mass m is calculated about an (infinitely inert) nucleus of load + Ze. The Hamiltonian function of the problem can be written in spatial polar coordinates r, ϑ, ψ and corresponding impulses.

$$H \equiv \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2}p_\vartheta^2 + \frac{1}{r^2\sin^2\vartheta}p_\psi^2\right) + V(r) = W. \qquad (1)$$

Central field potential V(r) will mean later

$$V(r) = -\frac{e^2 Z}{r}.$$

1) Cf. Ch 7, Sect. 5-7 of this volume.

The Hamiltonian partial differential equation

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial S}{\partial \vartheta}\right)^2+\frac{1}{r^2\sin^2\vartheta}\left(\frac{\partial S}{\partial \psi}\right)^2\right]+V(r)=W,$$

follows from (1) and can be separated in the following three partial equations

$$\frac{\partial S_\psi}{\partial \psi}=\alpha_\psi,$$

$$\left(\frac{\partial S_\vartheta}{\partial \vartheta}\right)^2+\frac{\alpha_\psi^2}{\sin^2\vartheta}=\alpha_\varphi^2,$$

$$\left(\frac{\partial S_r}{\partial r}\right)^2+\frac{\alpha_\varphi^2}{r^2}+2mV(r)=2mW$$

and

$$S=\int_{r_0}^{r}\frac{\partial S_r}{\partial r}\,dr+\int_{\vartheta_0}^{\vartheta}\frac{\partial S_\vartheta}{\partial \vartheta}\,d\vartheta+\int_{\psi_0}^{\psi}\frac{\partial S_\psi}{\partial \psi}\,d\psi.$$

The meaning of both of the first equations is, as is known, the theorem of the constancy of the impulse moment, first, its components in the direction of the arbritrarily assumed polar axis (Fig. 4)

$$p_\psi=mr^2\sin^2\vartheta\,\dot{\psi}=\alpha_\psi$$

and, second, its absolute amount $p_\varphi=\sqrt{p_\vartheta^2+\frac{p_\psi^2}{\sin^2\vartheta}}=mr^2\dot{\varphi}=\alpha_\varphi.$

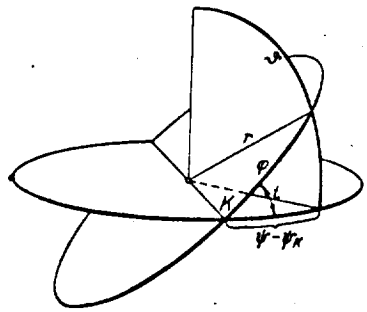

Fig. 4 Coordinates of the Keplerian motion

The third equation only reiterates the constancy of energy. The complete integral of (2):

$$S = \alpha_\psi(\psi - \psi_0) + \int_{\vartheta_0}^{\vartheta} \sqrt{\alpha_\varphi^2 - \frac{\alpha_\psi^2}{\sin^2\vartheta}}\, d\vartheta + \int_{r_0}^{r} \sqrt{2m[W - V(r)] - \frac{\alpha_\varphi^2}{r^2}}\, dr$$

arises from integration (4) in the form of S = S (r, ϑ, ψ, α_ψ, α_φ, W). It does not yet contain the action variables as arbitrary integration constants, but rather three parameters α_ψ, α_φ, W, which presented themselves without trouble during the integration. The lower limits of the integrals are related to a voluntary initial point, say to an arbitrarily positioned perihelion passage. The easiest is to think of these as ascending nodes. If one uses S in this form as the generatrix of a transformation (which happens frequently in celestial mechanics) the "canonical path elements" are obtained (cf. Ch. 3, Sect. 12 (7))

$$t + \beta_1 = \frac{\partial S}{\partial W} \qquad (-B_1 = \text{time of the perihelion passage})$$

$$\beta_2 = \frac{\partial S}{\partial \alpha_\varphi} \qquad (B_2 = \text{angular distance of the perihelion from ascending nodes}),$$

$$\beta_3 = \frac{\partial S}{\partial \alpha_\psi} \qquad (B_3 = \text{length of the ascending node}).$$

We do not use them, but first introduce the action variables J (Sect. 6 (1)) which are normalized as follows:

$$J_\psi = \oint \frac{\partial S}{\partial \psi}\, d\psi = 2\pi\alpha_\psi,$$

$$J_\vartheta = \oint \sqrt{\alpha_\varphi^2 - \frac{\alpha_\psi^2}{\sin^2\vartheta}}\, d\vartheta = 2\pi(\alpha_\varphi - \alpha_\psi),$$

$$J_r = \oint \sqrt{2m[W - V(r)] - \frac{\alpha_\varphi^2}{r^2}}\, dr,$$

from which

$$S = \frac{J_\psi}{2\pi}(\psi - \psi_0) + \left.\begin{aligned} &\int_{\vartheta_0}^{\vartheta} \sqrt{\frac{(J_\vartheta + J_\psi)^2}{4\pi^2} - \frac{J_\psi^2}{4\pi^2 \sin^2\vartheta}}\, d\vartheta \\ &+ \int_{r_0}^{r} \sqrt{2m[W(J_r, J_\vartheta, J_\psi) - V(r)] - \frac{(J_\vartheta + J_\psi)^2}{4\pi^2 r^2}}\, dr. \end{aligned}\right\} \quad (5)$$

From the defining equation for J_r, in which only both constants W and $\alpha_\varphi = \frac{J_\vartheta + J_\psi}{2\pi}$ still occur, one sees that even in the case of any given central field V(r) the path energy W can only depend on J_r and the sum $(J_\vartheta + J_\psi)$, not on J_ϑ and J_ψ alone. In this real decay of the system are expressed, namely the limitation to one path plane. If one selects $V(r) = -e^2Z/r$ as it corresponds to the Keplerian motion, the integral for Jr can be evaluated (for example on a complex path) and will produce

$$J_r = -J_\vartheta - J_\psi + 2\pi \frac{\sqrt{m}e^2 Z}{\sqrt{-2W}},$$

thus

$$W = -\frac{2\pi^2 m e^4 Z^2}{(J_r + J_\vartheta + J_\psi)^2}.$$

W is dependent in this case only on a linear combination of J; the system is "characteristically decomposed" twice, i.e. purely periodically (Sect. 7 and 8).

The angle variables are determined from

$$w_r = \frac{\partial S}{\partial J_r}, \qquad w_\vartheta = \frac{\partial S}{\partial J_\vartheta}, \qquad w_\psi = \frac{\partial S}{\partial J_\psi}.$$

Then in greater detail according to (5)

$$w_r = \int_{r_0}^{r} \frac{\partial^2 S}{\partial r \partial J_r} dr,$$

$$w_\vartheta = \int_{r_0}^{r} \frac{\partial^2 S}{\partial r \partial J_\vartheta} dr + \int_{\vartheta_0}^{\vartheta} \frac{(J_\vartheta + J_\psi) d\vartheta}{2\pi \sqrt{(J_\vartheta + J_\psi)^2 - \frac{J_\psi^2}{\sin^2 \vartheta}}},$$

$$w_\psi = \int_{r_0}^{r} \frac{\partial^2 S}{\partial r \partial J_\psi} dr + \int_{\vartheta_0}^{\vartheta} \frac{(J_\vartheta + J_\psi) d\vartheta}{2\pi \sqrt{(J_\vartheta + J_\psi)^2 - \frac{J_\psi^2}{\sin^2 \vartheta}}}$$

$$- \int_{\vartheta_0}^{\vartheta} \frac{J_\psi d\vartheta}{2\pi \sin^2 \vartheta \sqrt{(J_\vartheta + J_\psi)^2 - \frac{J_\psi^2}{\sin^2 \vartheta}}} + \frac{(\psi + \psi_0)}{2\pi}.$$

The two integrals to be directed over can be transformed with the help of the relationships

$$\cos i = \frac{J_\psi}{J_\vartheta + J_\psi}, \qquad \cos \vartheta = \sin i \sin \varphi, \qquad \sin(\psi - \psi_K) = \operatorname{cotan} \vartheta \operatorname{cotan} i,$$

which are found in Fig. 4. The first becomes equal to $(\varphi - \varphi_0)/2\pi$, the second to $(\psi - \psi_0')/2\pi$. Thereby φ_0 and φ signify the azimuths of the beginning and end positions of the electron, measured in the path plane from a fixed direction in them, 3.g. from the ascending nodes K. ψ_0 and ψ are the lengths of these points measured in the equatorial plane from an arbitrary direction fixed in them. However, one must be careful that ψ_0 means the space-bound "length" of the path point at time t_0. It can be different from ψ_0', that is, from the length of the initial position marked in the path plane at time t (and with it perhaps subjected to a precession). If one

directs the integrals across a complete libration of the r- coordinates, e.g. from one perihelion passage to the next, and takes into account that for this (Sect. 6 (4))

$$\oint_{r_0}^{r} \frac{\partial^2 S}{\partial r \partial J_r} dr = 1, \qquad \oint_{r_0}^{r} \frac{\partial^2 S}{\partial r \partial J_\vartheta} dr = \oint_{r_0}^{r} \frac{\partial^2 S}{\partial r \partial J_\psi} dr = 0,$$

one obtains the meaning of

$$\left.\begin{aligned} 2\pi w_1 = 2\pi w_r = {} & \begin{cases}\text{means anomaly measured from perihelion}\\ \text{position which is perhaps mobile in the}\\ \text{path plane;}\end{cases}\\ 2\pi w_2 = 2\pi(w_\vartheta - w_r) = {} & \begin{cases}\text{the azimuth of the perihelion set back}\\ \text{in the path plane;}\end{cases}\\ 2\pi w_3 = 2\pi(w_\psi - w_\vartheta) = {} & \begin{cases}\text{angle of precession of the path plane, or}\\ \text{the length of some location marked in it,}\\ \text{e.g. a node.}\end{cases}\end{aligned}\right\} \quad (6)$$

Naturally, for the Keplerian motion one must determine that the last two quantities are constants. Let us then again make a transformation to the quantities W_1, W_2, W_3 and the corresponding action variables. (In astronomy this corresponds to the transformation to the "Delaunayan path elements".) The generatrix is (cf. Sect. 7 (3))

$$S = J_1 w_r + J_2(w_\vartheta - w_r) + J_3(w_\psi - w_\vartheta),$$

from which

$$\begin{aligned} J_r &= J_1 - J_2, & J_1 &= J_r + J_\vartheta + J_\psi,\\ J_\vartheta &= J_2 - J_3, & J_2 &= J_\vartheta + J_\psi,\\ J_\psi &= J_3, & J_3 &= J_\psi \end{aligned}$$

and (cf. Sect. 8)

$$W = -\frac{2\pi^2 m e^4 Z^2}{J_1^2}. \quad (7)$$

In fact, $V_2 = V_3 = 0$; w_1, J_1 is the "real", w_2, J_2 and w_3, J_3 the "unreal" angle and action variables.

10. Definition of the angle and action variables for general multiple periodic systems. A system is termed r-fold periodic when its coordinates can be represented as r-fold Fourier series of time as in Sect. 6, Eq. (7):

$$q_k = \sum_{-\infty}^{+\infty} \cdots \sum_{-\infty}^{+\infty} a^{(k)}_{\tau_1 \ldots \tau_r} e^{2\pi i[(\tau_1 \nu_1 + \cdots \tau_f \nu_f) t + (\tau_1 \delta_1 + \cdots + \tau_f \delta_f)]}. \tag{1}$$

Thereby one first thinks of the cartesian coordinates of its element, but every coordinate system proceeding them and their corresponding impulses by unique transformation can be represented in this form. (For coordinates not uniquely determined several modifications are necessary, which can be drawn from the remarks in Sect. 5 and 6.)

All frequencies appearing in (1) are integral linear aggregates of r fundamental frequencies $\nu_1 \ldots \nu_r$, which we presuppose to be incommensurable since otherwise a representation with less than r fundamental frequencies is possible. The phases are just such combinations of the r quantities $\delta_\alpha (\alpha = 1,2 \ldots r)$. In order to satisfy an arbitrarily selected initial state, the amplitudes contain further 2 (f - r) constants C_m (m = 1,2,...2 (f - r)) and are, moreover, connected to each other by the laws of motion.

We introduce the new coordinate

$$w_\alpha = \nu_\alpha t + \delta_\alpha, \qquad (\alpha = 1, 2, \ldots r) \tag{2}$$

and obtain through this the reproduction of the q volume on the w volume, which is discussed in detail in Sect. 6 to 8:

$$\text{to which } \left.\begin{aligned} q_k &= q_k(\tilde{w}_1, \ldots \tilde{w}_r), \\ p_k &= p_k(\tilde{w}_1, \ldots \tilde{w}_r), \end{aligned}\right\} \tag{3}$$

is added. From this it follows that each unique function of p_k, q_k is also periodic in the w values. The variables (unknown to us) canonically conjugated with the w are termed J .

In any event, according to Ch. 3, Sect. 6 (3)

$$[w_\alpha, w_\beta] = 0.$$

For this one can also write

$$\frac{\partial}{\partial w_\alpha}\left(\sum_{j=1}^{f} p_j \frac{\partial q_j}{\partial w_\beta}\right) = \frac{\partial}{\partial w_\beta}\left(\sum_{j=1}^{f} p_j \frac{\partial q_j}{\partial w_\alpha}\right),$$

which shows that the expression

$$\sum_{\alpha=1}^{r}\left(\sum_{j=1}^{f} p_j \frac{\partial q_j}{\partial w_\alpha}\right) dw_\alpha$$

is a complete differential. The action integral

$$S = \int \sum_{j=1}^{f} p_j \, dq_j$$

is thus a function of its limits. (However, one must remember that the integration path is only free when R = f. If the system is decomposed, then it remains limited at the integral position c_m = constant.)

The integrals

$$J_\alpha = \int_{w_\alpha}^{w_\alpha + 1}\left(\sum_{1}^{f} p_j \frac{\partial q_j}{\partial w_\alpha}\right) dw_\alpha$$

which are directed from any given point of the w-axis across a straight unit distance R parallel to the axis are, as one can be easily convinced, independent of the initial point and thus constant along each mechanical path. They are the periodicity nodules of the action function

$$S = \sum_1^r \bar{J}_\alpha w_\alpha + F(\bar{J}_\alpha \bar{w}_\alpha).$$

written in the w variables.

If one performs with the help of (4) a second canonical transformation $(J, w) \rightarrow (\bar{J}, \bar{w})$, the w_α values result, which always increase linearly with time. This is true since the transformed Hamiltonian function can only depend on $\bar{J}_\alpha$ as a constant, since the system is to be considered as closed and as related to a static coordinate system, so that the $\dot{\bar{w}}_\alpha$ values are constant. From this, however, one sees that the function F is a constant and is actually independent of the quantities w_α. The transformation equations produce

$$J_\alpha = \partial S / \partial w_\alpha = \bar{J}_\alpha,$$

and the transformed Hamiltonian function is alone independent of J_α:

$$W = W(J_1, \cdots J_r). \tag{5}$$

The quantities w_α, J_α can thus justifiably be characterized as angle and action variables of the system (1).

The theorem

$$\Delta W = \sum_{\alpha} \frac{\partial W}{\partial J_\alpha} \Delta J_\alpha = \sum_{\alpha} \nu_\alpha \Delta J_\alpha \qquad (6)$$

follows from formula (5), as in Sect. 5 (7), for any two motions of the multiple periodic system determined by the neighboring value of the action variables. Here as previously it forms the basis of the Bohrian principle of correspondence.

The characteristics (2) to (5) are sufficient, according to a proof by F. Hund, to distinguish without arbitrariness canonically introduced variables as angle and action variables. They thereby remain undefined up to a linear integral transformation with the permanent one since the periodic return of the p- and q- values in the lattice points of the w value from which we begin still allows the free choice of the lattice cells. Thus, the angle variables as well as the action variables which are defined as integrals across the borders of the lattice cells and which are transformed contragradiently. One can find more on this subject in the book by Born[1]. The definitions given here are from W. Pauli.

11. The adiabatic invariants of the action variables. In the previous sections it was shown that a relatively simple

1) M. Born, Atommechanic I, Sect. 15, Berlin, 1925.

and clear formalism for the description of multiple periodic motions was gained with the introduction of angle and action variables. Its significance in quantum theory extends far beyond this.

This branch of physics assumes that not every physically conceivable state of the smallest electro-mechanic systems can exist in the world of atoms, but rather that there are distinct stationary conditions which almost appear alone. They must be characterized dynamically by certain constants in the motion of the mass points. If one wants to hold closely to the concepts of Bohrian atomic theory, one must ask what kind of quantities come under consideration for such a description. Since Planck's, Bohr's and Sommerfeld's statements it has been shown that the historical development in which "quantum condition" were coupled with the action variables, was no accident. They are so suitable, since, on the one hand, they are geometrically invariant, i.e. independent of the coordinates used for their derivation. This was shown by Brody's theorems which were discussed in Chapter 3, Sect. 5[1].

Secondly, they are "adiabatically invariant." One understands by this the following very important characteristic: Let the motion state of the mechanical system depend on certain continuous changeable system parameters a_k aside from p_j, q_j (one considers the

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1) E. Brody, ZS f. Phys. Vol. 6, p. 224, 1921.

attractive force of the sun on the planets as changeable, for example), but so that for every fixed value for a_k the motion remains conditionally periodic and possesses the same degree of decomposition. The motion integrals are then dependent generally also on the parameters a_k, thus are of form $F(p_j, q_j, a_k) = \alpha(a_k)$. There are, however, certain functions of F--thus also integrals--which are not affected by a change in a_k in the first approximation, but instead also remain constants of motion when one conceives of the parameter a_k as being invariable, insofar as its variation only is accomplished slowly, so that it is imperceptibly small during a quasi-period of the fixed system. These integrals alone are the action variables. For this reason they are especially suitable for describing the "states" of atoms subjected to many perturbations. As understood by quantum theory their condition also remains "stationary" with sufficient slow perturbations, and one can imagine that the extraordinary stability of the atoms with respect to the slow influences is associated with the constancy of the action variables. The effectiveness of sudden influences (shocks, radiation) cannot yet be explained.

The thought process of the proof for the adiabatic invariants of the action variables is as follows: Let the Hamiltonian function of the system be, aside from p and q, dependent on the parameters a (t) which are variable with time. One follows the influence of their variants in the limit $\dot{a} \to 0$ and simultaneously expands the

calculation over a time T so that the integral $\int_T \dot{a}\,dt$ contains the finite valued. In any point in time the motion will not be very different from those motions which are additionally periodic and which would appear for a constant a. For the latter one could introduce angle and action variables with the generatrices, S(q, J, a) according to the manner shown in Sect. 4, 5 or 6. One now uses this same transformation, although a varies and J_k is not completely constant, but rather procedes from the canonical equations

$$\dot{w}_k = \frac{\partial K}{\partial J_k}, \qquad \dot{J}_k = -\frac{\partial K}{\partial w_k}$$

variably. According to Ch. 3, Sect. 3, Eq. 6, the transformed Hamiltonian function K is

$$K = W(J_1, \ldots J_s, a) + \frac{\partial S[q(Jw)Ja]}{\partial a}\dot{a},$$

so that

$$\dot{J}_k = -\frac{\partial}{\partial w_k}\left(\frac{\partial S}{\partial a}\right)\dot{a}. \tag{1}$$

One must now presuppose that the variation of a occurs without being related to one of the motion frequencies--the simplest would be to assume that this occurs uniquely. Then, principally for frequencies $(\tau\nu) = \tau_1\nu_1 + \cdots + \tau_s\nu_s$, which are independent of a and not equal to 0,

$$\lim_{\dot{a}\to 0}\int_T \dot{a}\,e^{2\pi i[(\tau\nu)t+(\tau\delta)]}\,dt = \lim \dot{a}\,\frac{e^{2\pi i[(\tau\nu)t+(\tau\delta)]}}{2\pi i(\tau\nu)} = 0. \tag{2}$$

This also holds true for frequencies which are dependent on the integral. The integral can be divided into partial integrals over

quasi- periods and in every section the integrand can be expanded by t. Then the first expansion numbers again give the expression (2), and it can be shown of the others that altogether they amount to nothing in the limit case $\dot{a} \to 0$.

Now in formula (1) the expression $\partial/\partial w_k(\partial S/\partial a)$ is to be thought of as a Fourier series without a constant number as long as it is derived through a characteristic angle variable. This is true since the derivation has eliminated all the numbers independent of w_k so that only numbers remain whose frequency $(\tau\nu)$ contains a portion $\tau_k\nu_k$ with non-vanishing τ_k. If one thus forms

$$\int_T \dot{J}_k\,dt = -\int_T \dot{a}\,\frac{\partial}{\partial w_k}\left(\frac{\partial S}{\partial a}\right)dt,$$

one may include

$$\int_T \dot{J}_k\,dt = 0,$$

if in the course of time T one of the frequencies $(\tau\nu)$ does not pass the 0 volume, i.e., is the system procedes through a state of further decomposition.[1] Under closer observation this is of course such a serious limitation that the practical value of the calculation would be small if one were not able to be, by and large, rid of it. Since, as was shown in Sect. 8, the positions of random decomposition

1) The original proofs up to this point exist in P. Ehrenfest, Ann. d. Phys., Vol. 51, p. 327, 1916; J. M. Burgers, his dissertation cited in Sect. 1 and Ann. d. Phys., Vol. 52, p. 195, 1917; G. Krutkow, Amst. Verl., Vol. 27, p. 908, 1918; cf. also the summary by P. Ehrenfest in Naturwissenschaft, Vol. 11, p. 543, 1923

are even closely situated with continual changes of the frequencies. Von Laue[2] has shown though that the constancy of J_k also is presumed when one of the frequencies ($\tau\nu$) vanishes but not more than a power of λ. This proves the adiabatic invariants of the action variables for most cases. The proof does not extend to uncharacteristic action variables. This is, however, irrelevant in quantum theory since they do not influence the system energy. Moreover, he makes clear that the invariance stops when the "adiabatic transformation" is directed across a finite distance far beyond a condition of decomposition.

Examples of adiabatic transformations: a string pendulum whose string length is gradually shortened. An oscillating string which is gradually shortened by directing an elastic tube from one end toward the other. A plane oscillator whose potential ellipse is slowly deformed or rotated and so on. The last case can be constructed easily so that the invariance of the action variable stops: during the transformation of the axes of the potential ellipse, one can stop at one at which both frequencies (the x and y waves) are commensurable. If a finite section is rotated in this deomposition state, and if the binding forces are further changed, then the J values are changed.[1]

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1) Further references and examples are found in P. Ehrenfest, l.c.; N. Bohr, Quantentheorie (cf. footnote in Sect. 1).

III. METHODS OF PERTURBATION CALCULATION FOR THE HAMILTIONIAN FUNCTION INDEPENDENT OF TIME

12. Preliminary remarks. Before a description of systematic perturbation calculation is given in Sections III and IV, one would do well to remember that in many cases in which one only wants to obtain a limited goal simpler methods of calculation can fulfil this goal. Much use will be made, however, of theorems on perturbation calculation for this (e.g., of the the theorem that the mean value of the perturbation energy of the first order is a constant across the undisturbed motion, etc.). Thus, several perturbation problems for Kaplerian motion can be eliminated in the first approximation in an elementary manner; cf. the calculations by Bohr[1], Lenz[2] and Klein[3] on the hydrogen atom in exterior fields.

Furthermore, the calculation of energy perturbation in the first and second approximation is accomplished with the help of the adiabatic method without actually using perturbation calculations, i.e. by drawing in theorems derived in Section 11. The procedure was used by Kramers[4] and by Shrödinger[5] and was proven

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1) N. Bohr, Quantentheorie.

2) W. Lenz, ZS f. Phys. Vol. 24, p. 197, 1924.

3) O. Klein, ZS f. Phys. Vol. 22, p. 109, 1924

4) A. H. Kramers, Dissert. 1919; Copenhagen Academy, Vol. 8, III 1919; ZS f. Phys. Vol. 13, p. 312, 1919

5) E. Schrödinger, ZS f. Phys. Vol. 11, p. 172, 1922.

by the latter in a simple fashion. The systematic procedure is to be preferred for the calculation of higher approximations.

13. The Semiconvergent character of perturbation calculation. After having developed the formal system for the description of multiple periodic motions in Section II, we might ask whether or not more general motions can be approximated in many cases by conditionally periodic motions. The proxess which has been developed for a long time in celestial mechanics under the name of perturbation calculation has precisely this goal.

The pure Kepler motion of a planet is, for example "perturbed" bu the presence of a second planet. The complicated motion which occurs can be considered in every time element as a part of a suitably selective Kepler motion; but, however, its path elements will change with time. If one now views these quantities, which were earlier fixed, as coordinates, one is making analytical use of Lagrange's method for the variation of the constants[1].

This becomes itself an approximation procedure if one considers that the forces emanating from the disturbing planets usually (i.e., in a certain region G of the coordinates) are small with respect to the attraction by the sun in the proportion λ of the masses of both bodies. Analytically speaking, this fact corresponds

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1) H. Poincaré, Leçons etc. (cf. footnote from Sect. 1) Vol. I, Ch. IV and V.

to a development of the differential equations of motion by powers of λ. According to a theorem by Poincaré, the possibility of a similar expansion of integrals follows from this[1].

Meanwhile, the success of this procedure is dependent on two things: First, one must always consider whether the integrated motion also actually remains in the region G, so that the presupposition of the series expansion remains in force. Second, one will only be able to obtain a complete (i.e., convergent) termination of the disturbed motion when it itself is conditionally periodic. Thus, the question of the existence of unique integrals (the action variables) for the perturbed motion arises. The proof of their non-existence can be made in certain cases according to a method by Poincaré (these cases are in fact the rule). How will this inner contradiction express itself in this approximation? It will occur in a two-fold manner. Either the differential equations will not permit formal integration at a definite point in the expansion (the calculation can simply not be done then). Or--this leads to the peculiarity of perturbation calculation--one can formally produce the description by angle and action variables, the conditionally periodic representation, by applying the coordinates as multiple Fourier series and determining their coefficients according to a

1) H. Poincaré, Methodes nouvelles, Vol. II, Ch. VII and XIII.

certain calculation pattern. Then, however, the series obtained will not converge. In spite of this they have acquired the greatest practical significance for celestial mechanics and physics because of their semi-convergence, which permits one to calculate with them as with convergent series. The evaluation of the error permitted here requires convergence investigations, for which one can apply references in Poincare[13].

14. The arbitrary, multiple periodic statement for the perturbed motion. Let

$$H \equiv H_1 + H_2 = W$$

be the Hamiltonian function of a mechanical system, a total system and

$$H_1 = W_1$$

that of a partial system, whose conditionally periodic and thereby s-fold motion is drawn. It is described after (according to Sect. 6 or 10) one has introduced the angle and action variables w_k^o , J_k^o of the partial system with the help of the transformation equations

$$\left.\begin{aligned} q_j &= \sum_{-\infty}^{+\infty} \cdots \sum_{-\infty}^{+\infty} a^{(j)}_{\tau_1 \ldots \tau_f} e^{2\pi i (\tau_1 w_1^o + \cdots + \tau_f w_f^o)}, \\ p_j &= \sum_{-\infty}^{+\infty} \cdots \sum_{-\infty}^{+\infty} b^{(j)}_{\tau_1 \ldots \tau_f} e^{2\pi i (\tau_1 w_1^o + \cdots + \tau_f w_f^o)}, \end{aligned}\right\} \tag{1}$$

by the conditions of time

$$\begin{aligned} J_k^o &= \text{constant} \quad (k = 1, 2, \ldots, f) \\ w_\alpha^o &= \frac{\partial H_1}{\partial J_\alpha^o} t + \delta_\alpha^o, \quad (\alpha = 1, 2, \ldots, f - s, && \text{Characteristic variable} \\ w_\varrho^o &= \text{constant} \quad (\varrho = f - s + 1, \ldots, f, && \text{Uncharacteristic variable} \end{aligned}$$

The transformation equations (1) contain a canonical transformation independent of every definite motion problem (c.f. Ch. 3, Sect. 3). The variables w_k^o, J_k^o, are also canonical in relationship to the Hamiltonian function

$$H \equiv H_1(J_\alpha^0) + H_2(J_\alpha^0, J_\varrho^0, w_\alpha^0, w_\varrho^0) = W,$$

whose first member H_1 can only be dependent on the J_α^o values according to our presuppositions. Actually, in the motion of the total system they are no longer constant or linear in time, but, on the other hand, the quantities $J_\kappa^o, w_\varrho^o, \nu_\alpha^o = \partial H/\partial J_\alpha^o, \delta_\alpha^o$ change in some fashion. The laws for their variation are the canonical equations written in terms of w_k^o and J_k^o.

Then

$$\dot{w}_k^0 = \frac{\partial H}{\partial J_k^0}, \qquad \dot{J}_k^0 = -\frac{\partial H}{\partial w_k^0}.$$

It will not be superfluous to spend a moment with the new significance of the transformation equations (1). By formally permitting the Fourier series to exist one allows the path region of the q volume to reproduce itself on the unit cube of the w^o volume and allows this image to repeat itself periodically in the w^o volume (cf. Section 7). However, because of the variability of the amplitudes in the Fourier series (1) one must consider also that the path region in the q volume has changed, perhaps is no longer even fixed. The equations (1) are still only uncharacteristic Fourier series. With the linearity of w^o in time and with the decay of w^o the progress of the path curve in the w^o line stops completely. Nevertheless, something can be said about the

change in w_k^o without calculation if one assumes that the motion of the total system is conditionally periodic. For this motion there are new angle and action variables w_k and J_k which are related to the system position by characteristic Fourier series.

$$\left.\begin{aligned} q_j &= \sum_{-\infty}^{+\infty} \cdots \sum_{-\infty}^{+\infty} A^{(j)}_{\tau_1 \ldots \tau_f} e^{2\pi i(\tau_1 w_1 + \cdots + \tau_f w_f)}, \\ p_j &= \sum_{-\infty}^{+\infty} \cdots \sum_{-\infty}^{+\infty} B^{(j)}_{\tau_1 \ldots \tau_f} e^{2\pi i(\tau_1 w_1 + \cdots + \tau_f w_f)} \end{aligned}\right\}$$

We now trace in the q and w^o volume and in the w volume a definite motion of the system which we direct voluntarily along a straight unit distance parallel to the w_j axis in the w volume. Thereby, the system in the pq volume returned to the initial state on a certain curve. In the equations (1) the left sides and on the right the amplitudes a, b (which are also multiple periodical in w as unique functions of p, q) have reached the old value. The arguments w_k^o have thus certainly attained values which differ from their initial values by 0 or some whole numbers. If one now still assumes that the influence of the perturbation on all w_k^o is small with respect to their Eigen motion in the unperturbed case, it follows that w_j^o has also increased by 1, but that the other w_1^o values have not increased. This means the same thing as the relationship between new and old angle variables

$$w_k = w_k^0 + (\tilde{w}_1^0, \ldots \tilde{w}_f^0),$$

in which only the coefficients of a Fourier series are still undefined.

Two presuppositions are needed for the proof of this conclusion: First, that the perturbed system is also multiple periodic by setting up the same series (2) without testing, by postulating the equations (3) and by determining its Fourier coefficients by a formal process, one produces the multiple periodic representation for any given perturbed systems in perturbation calculation. However, this path is not always available.

Equation (3) is based on the further assumption that the perturbation influence on w_k^0 is small with respect to its Eigen motion in the unperturbed case. This assumption is impossible for all earlier decayed (i.e. constant w_k^0. Equation (3) cannot be postulated for it, but one must strictly decide by integration whether the degrees of freedom one is concerned with definitely behave periodically after perturbation. With every perturbation calculation which depends on the reduction of the decay in general a partial differential equation thus appears which is so complicated that its integrability is questionable.

15. Development of integrals according to powers of a parameter; intermediate motions. Let the Hamiltonian function of the mechanical system be expanded according to powers of a small parameter λ

$$H(p,q) \equiv H_0 + \lambda H_1 + \lambda^2 H_2 + \cdots = W$$

and with it the canonical equations

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k}.$$

$p_k^0(t),\ q_k^0(t)$ of the partial problem

$$H_0 = W_0; \qquad \dot{q}_k^0 = \frac{\partial H_0}{\partial p_k^0}, \quad \dot{p}_k^0 = -\frac{\partial H_0}{\partial q_k^0},$$

and if the functions H_1, H_2 etc. are expandable according to powers of $(p_k - p_k^0),\ (q_k - q_k^0)$ for all values of the same, the series expansions can be found for p_k, q_k according to powers of λ

$$\left.\begin{aligned} q_k &= q_k^0 + \lambda q_k' + \lambda^2 q_k'' + \cdots, \\ p_k &= p_k^0 + \lambda p_k' + \lambda^2 p_k'' + \cdots, \end{aligned}\right\}$$

which formally satisfy the differential equation (2). By introducing (4) into (2) and rearrangement one obtains

$$\left.\begin{aligned} &\dot{q}_k^0 + \lambda \dot{q}_k' + \lambda^2 \dot{q}_k'' + \cdots \\ &\quad = \frac{\partial H_0}{\partial p_k^0} + \lambda \left\{ \frac{\partial H_1}{\partial p_k^0} + \sum_i \frac{\partial^2 H_0}{\partial q_i^0 \partial p_k^0} q_i' + \sum_i \frac{\partial^2 H_0}{\partial p_i^0 \partial p_k^0} p_i' \right\} \\ &\qquad + \lambda^2 \{\cdots\} + \cdots \end{aligned}\right\}$$

and similarly construction equations for p_k. Thereby $\partial H_0/\partial p_k^0$ is used as an abbreviation $\partial H_0/\partial p_k$, in which q_k^0, p_k^0 replaces q_k, p_k with an unchanged form of the function, etc.

If one concludes from (5) that individually the coefficients are (5) that individually the coefficients are equal to the same powers of λ on the left and right a series of linear differential equations for the determination of the unknown $q_k^{(m)}$, $p_k^{(m)}$ result in place of (2) excepting (3). Poincaré has proven that the process converges as long as the motion does not extend past the validity limits set by the presuppositions (expandability of H by λ and by $p - p^0$, $q - q^0$).

One notices that a certain arbitrariness is associated with

the procedure from (5) that like powers of λ are equal; one does not need to view them as being exactly equal but rather only as far as members of the next smallest order. To express this in another way: Before equating the members which have the same order one can perform a certain rearrangement of the series, and, for example, rewrite the series (1) as follows after breaking down H_n into $H'_n + H''_n$:

$$\left.\begin{aligned} H &\equiv (H_0 + \lambda H_1') + \lambda(H_1' + \lambda H_2') + \lambda^2(H_2'' + \lambda H_3') + \cdots \\ &\equiv H_0^* + \lambda H_1^* + \lambda^2 H_2^* + \cdots \end{aligned}\right\}$$

This does not change the motion problem of the total system but does open up a new approximation procedure.

What is the physical meaning of this indetermination? The entire approximation procedure means that the motion problems are solved one by one by means of the Hamiltonian functions H_o, $H_o + \lambda H_1$ etc., which are not the same as the total aberrations, which become smaller and smaller. This is called the introduction of the intermediary motions. The first intermediary motion is characterized by the requirement that it can only deviate from the total motion by member $\sim\lambda$, the second only by $\sim\lambda^2$ etc. Of course, a certain arbitrariness always remains, and one sets up the calculation so that it becomes as formally simple as possible.

16. Poincare's proof for the non-existence of unique integrals.[1]

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[1] Strictly taken, the mathematical minimal postulate is somewhat less (cf. Poincare elsewhere); but is almost always fulfilled when unique integrals for the disturbed motion actually exist.

Let the Hamiltonian function of a mechanical system be

$$H \equiv H_0 + \lambda H_1 + \lambda^2 H_2 + \cdots$$

and H = W be an integral of motion. In addition let F = α be a further integral independent of it. According to the theorem by Poisson (Chapter 3, Sect. 7 (5)) the condition (H, F) = 0 must be sufficient. (The meaning is simple--the gradient of F stands perpendicular to the phase path element.) On the other hand, when F = α is unique in the regions observed, a variation of the theorem from Sect. 15 which says that F can be expanded by powers of α, can become

$$F \equiv F_0 + \lambda F_1 + \lambda^2 F_2 + \cdots$$

For every unique integral the equation

$$\left.\begin{aligned} 0 = (H, F) \equiv & \sum_k \left\{ \frac{\partial H_0}{\partial p_k} \frac{\partial F_0}{\partial q_k} - \frac{\partial H_0}{\partial q_k} \frac{\partial F_0}{\partial p_k} \right\} \\ & + \lambda \sum_k \left\{ \frac{\partial H_0}{\partial p_k} \frac{\partial F_1}{\partial q_k} + \frac{\partial H_1}{\partial p_k} \frac{\partial F_0}{\partial q_k} - \frac{\partial H_0}{\partial q_k} \frac{\partial F_1}{\partial p_k} - \frac{\partial H_1}{\partial q_k} \frac{\partial F_0}{\partial p_k} \right\} + \cdots \end{aligned}\right\}$$

holds. If it cannot be fulfilled by any function F, then no unique integral exists independent of H. It can only be fulfilled when the expressions in parentheses alone vanish.

One can presuppose (proof by Poincare) that F_0 was independent from H_0, i.e., that an independent integral does not first arise through perturbation. Moreover, it is purposeful to consider angle and action variables of the unperturbed problem w, J as introduced for further calculation. Then H_0 and also F_0 on account of the disappearance of the first parentheses, are also dependent

only on J_k. For what follows it is necessary to differentiate between two cases:

Case 1. The perturbed system is not decomposed. The second parentheses to 0 reads to the equation

$$\sum_k \left(\frac{\partial H_0}{\partial J_k} \frac{\partial F_1}{\partial w_k} - \frac{\partial H_1}{\partial w_k} \frac{\partial F_0}{\partial J_k} \right) = 0.$$

If one considers H_1 and F_1 as expanded in Fourier series of W_k,

$$H_1 = \sum \cdots \sum B_{\tau_1 \ldots \tau_f} e^{2\pi i (\tau_1 w_1 + \cdots + \tau_f w_f)}, \qquad F_1 = \sum \cdots \sum b_{\tau_1 \ldots \tau_f} e^{2\pi i (\tau_1 w_1 + \cdots + \tau_f w_f)},$$

is follows that

$$(\tau_1 \nu_1 + \cdots \tau_f \nu_f) b_{\tau_1 \ldots \tau_f} = \left(\sum_k \tau_k \frac{\partial F_0}{\partial J_k} \right) B_{\tau_1 \ldots \tau_f}.$$

In this equation the $\nu_k = \partial H_0 / \partial J_k$, the $\partial F_0 / \partial J_k$, the b_τ and B_τ are dependent on the J_k values. It could serve to determine the quantities b_τ as functions of J_k from the known Fourier coefficients of the perturbation function of the first order H_1. If J_k changes continually during variation of the initial motion, for infinitely many values, one of sums $\sum \tau_\kappa \nu_\kappa$ will vanish (cf. Sect. 8 on random decomposition). Moreover, each time an entire class of them, rather than an individual, namely also $\sum \tau'_\kappa \nu_\kappa$, insofar as $\tau'_k = m\tau_k$; in fact, an entire family of classes which also contains all standing systems τ_k not in the same relationship, for which the sum $\sum \tau_k \nu_k = 0$ for fixed ν_k. The corresponding coefficients $B_{\tau_1 \ldots \tau_f}$ can be described as the coefficients which become secular

in a sense which will only become clear after analyses from Sect. 17. In order that equation (2) can exist at all for any given values J_k, it is necessary that all B_τ which are becoming secular vanish[1]. B_τ is given by the mechanical problem and in general does not fulfil such conditions. Thus, Poincare calls families of B_τ which are becoming secular and which do not fulfill the condition, regular families and those which fulfill the condition, singular. In the region of the J volume in which points with regular families are densely situated, no unique integral of the perturbed motion can exist aside from H = W.

Case 2. If the unperturbed system is actually S-fold decomposed, the condition of non-existance takes on a different character. The requirement corresponding to equation (2) becomes

$$2\pi i\left[\left(\sum \tau_\alpha \nu_\alpha\right) b_{\tau_\alpha} - \left(\sum \tau_\alpha \frac{\partial F_0}{\partial J_\alpha}\right) B_{\tau_\alpha}\right] + \sum_\varrho \left[\frac{\partial B_{\tau_\alpha}}{\partial J_\varrho}\frac{\partial F_0}{\partial w_\varrho} - \frac{\partial B_{\tau_\alpha}}{\partial w_\varrho}\frac{\partial F_0}{\partial J_\varrho}\right] = 0.$$

One cannot, thus, demand as was previously the case a simple vanishing of the (complex) B_τ at the points of random decomposition. The calculation which we do not reproduce here in detail leads to the formulation below for two non-decomposed degrees of freedom (e.g. two planets). When the B_τ', B_τ''... of a class, which can also be described $B_{(n\tau)}$, $B_{(m\tau)}$..., are created so that all products $(B_{n\tau})^m \cdot (B_{m\tau})^{-n}$ of this class are only dependent on 2s $-\mu$ of the f + s variables J_α, J_ϱ, w_ϱ, the class is called singular of the

μ-th order. In a region of the J volume in which points are situated densely, whose corresponding classes of $B_{\tau\sigma}$ are only singular of the μ-th order at most μ independent unique integrals of the perturbed motion can exist aside from H = W.

The Poincarian proof of the non-existence of a fifth unique integral in the problem of three bodies (which we consider to be related to the mass point so that the sixth mass point are not considered (is based on such an observation). If one represents the positions of each of the two planets the purely periodic function of its mean anomoly w' or w'' (cf. Sect. 9) as is calculated for vanishing simulaneous influence, the perturbation function becomes a two-fold Fourier series in w' and w'':

$$H_1 = \sum\sum B_{\tau'\tau''} e^{2\pi i(\tau' w' + \tau'' w'')}.$$

Of the products $(B_{n\tau'n\tau''})^m \cdot (B_{m\tau'm\tau''})^{-n}$ of a class always six, but not five, are coupled by a functional relationship as Poincare has shown. Thus, only 5 = 2s - μ = 8 - 3 are independent of each other. Therefore, aside from the energy integral there exist still three further unique integrals--the surface theorems--and no more.

17. The Method of Secular Perturbations[1]. This is a predecessor of the later complete expansions which has become famous by its astronomical applications. It was the first to be translated into atomic mechanics by himself in his Copenhagen Academy

1) Cf. Poincare, Lecons etc. (cf. footnote from Sect. 1) Ch. VIII and IX.

Studies[1]. Although actually only a part of a comprensive method for calculating the perturbation in initial systems actually decomposed cf. Sect. 19), in most literature, it is usually abbreviated and not presented in the strict form of Sect. 15. We will now reproduce it in the usual way.

It is useful to present a certain graphic image for the calculation, as the simplest example, the perturbation of the Kepler Path of an electron through a constant homogeneous force field ("Stark" effect). The Hamiltonian Function written in the angle variables in the Kepler Path has the form

$$H \equiv H_0(J_1^0) + \lambda H_1(J_1^0, J_\varrho^0, w_1^0, w_\varrho^0) = W; \qquad (1)$$

w_1^0 is the mean anomaly of the planet according to Sect. 9, Equation (6), J_1^0 the conjugated real action variable (proportional to the root from the large axis of the path) appearing in H_0 according to Sect. 9, Equation (7); w_ϱ^0, J_ϱ^0, the unreal variables, are the azimuth of the perihelion and the precession angle of the plane. Since the transformations $(p,q) \rightarrow (J^0, w^0)$ are given by Fourier series of w_1^0, we have to imagine H_1 as the periodic function of this quantity. For instance, H_1 means the potential of the electron and the outer field F during the "Stark" effect. If its force lines fall in the direction of the polar axis (z-Axis), then $H_1 = -eFz$, wherein the z-coordinates of the electron revolving in the path are represented in the unperturbed case by a characteristic Fourier series in w_1 in whose amplitudes the remaining

1) N. Bohr, Quantentheorie (cf. footnote on Ch. 1)

(constant) path elements appear. The parameter indicates in the general case the relationship of the perturbing forces to the inner forces of the unperturbed system and is small in respect to one.

From the canonical equations

$$\left.\begin{aligned} \dot{u}_1^0 &= \frac{\partial H_0}{\partial J_1^0} + \lambda \frac{\partial H_1}{\partial J_1^0}, \qquad \dot{w}_\varrho^0 = \lambda \frac{\partial H_1}{\partial J_\varrho^0}, \\ \dot{J}_1^0 &= -\lambda \frac{\partial H_1}{\partial u_1^0}, \qquad \dot{J}_\varrho^0 = -\lambda \frac{\partial H_1}{\partial w_\varrho^0} \end{aligned}\right\} \tag{2}$$

one sees that the change in the previous constant path elements proceeds very slowly while the frequency of the unperturbed path revolution $v_1^0 = \partial H_0 / \partial J_1^0$ is of Order 1.

One can divide the notion into sections which are given by the growth of w_1^0 always by one unit, and the time segments T', T",... corresponding to the approximate size of a period T of the unperturbed path revolution. However, one can further resolve the rate of change $\dot{x}$ given in (2) of a path element x in every time segment $T^{(n)}$ into a mean rate of change--we will indicate it with Dx/Dt--and the deviation from it. Then, Dx/Dt will be reliable for nearly uniform perturbations of the elements, which are in fact small in the individual segment and of order λ, but which can amount to finite sums in the course of many segments; one calls them secular perturbations. Short periodical oscillations will rest over this, approximately in the rhythm of the undisturbed path revolution, and they will never exceed the quantitative order λ.

We now turn aside from them since in the next section a general procedure for calculation of such short periodic perturbations will be described.

In order to calculate secular perturbations, we have to average the equations (2) over a section $T^{(n)}$. At the same time the mean time of $\partial H_1/\partial x$ appears on the right. One can be easily convinced that it differs from chronologically linear growth only by quantities of order λ^2 because of the small deviations of J_k^0, w_ρ^0 from the constancy and the quantity w_1^0, from the value $\partial \bar{H}_1/\partial x_1$, understood to be under $\bar{H}_1$ the spatial mean value of H_1 across the unit distance of w_1^0 or the mean time over a period of that unperturbed path which directly oscillates the actual path. Accordingly, if one inserts the mean value $\bar{H}_1(J_1^0, J_\rho^0, w_\rho^0)$ in place of H_1 in (2), the original step-curve like definition of Dx/Dt is substituted by a uniform definition. From Equations (2), since $\bar{H}_1$ no longer depends on w_1^0

$$\left.\begin{aligned} \frac{Dw_1^0}{Dt} &= \nu_1^0 + \lambda \frac{\partial \bar{H}_1}{\partial J_1^0} + \lambda^2 \ldots, & \frac{Dw_\varrho^0}{Dt} &= \lambda \frac{\partial \bar{H}_1}{\partial J_\varrho^0} + \lambda^2 \ldots, \\ \frac{DJ_1^0}{Dt} &= 0 + \lambda^2 \ldots, & \frac{DJ_\varrho^0}{Dt} &= -\lambda \frac{\partial \bar{H}_1}{\partial w_\varrho^0} + \lambda^2 \ldots, \end{aligned}\right\} \tag{3}$$

and the equations lead to the further conclusions that

$$\frac{D\bar{H}_1}{Dt} = \frac{\partial \bar{H}_1}{\partial J_1^0}\frac{DJ_1^0}{Dt} + \sum_\varrho \frac{\partial \bar{H}_1}{\partial w_\varrho^0}\frac{Dw_\varrho^0}{Dt} + \sum_\varrho \frac{\partial \bar{H}_1}{\partial J_\varrho^0}\frac{DJ_\varrho^0}{Dt} = 0 + \lambda^2 \ldots \tag{4}$$

one discovered that both quantities J_1^0 and $\bar{H}_1$ only vary secularly

with speeds $\sim \lambda^2$. Thus, in times of quantitative order T/λ they only increase by $\sim \lambda$, and this remains true when one also assumes that J_1^0 is subject to short periodical fluctuations. In such times, on the other hand, the w_ρ^0 , J_ρ^0 are subject to finite growth.

If one further assumes that the motion of the system remains periodic or conditionally periodic with the inclusion of the perturbations, the J_κ^0, w_ρ^0 values return to their initial value individually in intervals of order T/λ , and one can conclude that J_1^0 and $\overline{H}_1^0$ are not only constant in such time segments, but are continually constant up to fluctuations of order λ . (In the case of conditional periodicity this conclusion does not seem to be absolutely mandetory since a quasi-period then has order $T/\lambda^{(f-1)}$ but this objection disappears under closer consideration, which would be too involved here.)

The continual constancy of J_1^0 (thus the large axis of the ellipse) is the first premise which the famous Laplacian proof for the stability of the planetary system is based on. We see here that it is indeed quite right to begin from the postulate of the conditionally periodic total motion. This characteristic is in no way proven; on the contrary, the result of Poincare's investigations (see Sect. 16) is that there is no conditionally periodic motion present even with the three body problem. This is the reason why the Laplacian stability proof can no longer be taken as a true proof; it only shows the constancy of the large axis for long, but not arbitrarily long times. The second half of the "Laplacian proof" which bases the constant smallness of the eccentricities and slopes

on the approximated constancy of the large path axes, stands and falls with the former.

For our purposes we have acquired the fact that, presupposing conditionally periodic total motion, the mean perturbation potential $\bar{H}_1$ up to quantities $\sim \lambda$ are set constant and the variable J_1^0 in it can be viewed as a constant. This reduces the secular equations (3) of the J_ϱ^0 , w_ϱ^0 in the first approximation to a motion problem s = f-1 degrees of freedom. (s, as previously, is the degree of decomposition), which is given by the canonical equations

$$\frac{Dw_\varrho^0}{Dt} = \frac{\partial \lambda \bar{H}_1}{\partial J_\varrho^0}, \qquad \frac{DJ_\varrho^0}{Dt} = -\frac{\partial \lambda H_1}{\partial w_\varrho^0} \tag{4}$$

and the "energy equation"

$$\lambda \bar{H}_1(J_\varrho^0, w_\varrho^0) = \lambda W_1.$$

if it is possible to integrate the problem, for example, over the. Hamiltonian partial differential equation

$$\bar{H}_1\left(\frac{\partial S}{\partial w_\varrho^0}, w_\varrho^0\right) = W_1$$

this finally justifies the supposition on the conditionally periodic character. One can reduce the angle and action variables w_ϱ , J_ϱ in such a manner that W_1 alone becomes a function of the J_ϱ values in that the w_ϱ^0, J_ϱ^0 are represented as periodic functions of the w_ϱ values which increase linearly with time.

18. Perturbation of a non-decomposed system. We now turn to a presentation of the current form of the perturbation calculations, a logical completion of the expansion by a parameter λ [1].

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[1] Cf. H. Poincaré, Methodes nouvelles, Vol II, Ch. 9, M. Born, and W. Pauli, Jr. 25 f. Phys. Vol. 10, p. 137. 1922.

The most simple case which can appear is that of a conditionally periodic, non decomposed initial system in whose angle and action variables w_k^0 , J_k^0 state the problem:

$$H = H_0(J^0) + \lambda H_1(J^0, w^0) + \cdots + \lambda^n H_n(J^0, w^0) + \cdots = W. \qquad (1)$$

under the arbitrary assumption that the perturbed system is also conditionally periodic we look for the new angle and action variables w_k, J_k. After their introduction H alone must be a function W(J) of the J_k values.

As always the Jacobian method serves us as the integration procedure[1]. Thus, we determine from the Hamiltonian partial differential equation

$$H\left(\frac{\partial S}{\partial w^0}, w^0\right) = W$$

a function $S(w_k^0, J_k)$ the genereratrics of the transformation

$$J_k^0 = \frac{\partial S}{\partial w_k^0}, \qquad w_k = \frac{\partial S}{\partial J_k}. \qquad (2)$$

According to the theorem by Poincare of (Section 15) the determination of the J_k, w_k values has the form of a power expansion by ; We thus state:

$$S = S_0 + \lambda S_1 + \lambda^2 S_2 + \cdots + \lambda^n S_n + \cdots.$$

since according to the presupposition none of the old angle variables is decomposed during the unperturbed motion, but rather that all w_k^0 values have finite rates of change $\dot{w}_k^0 = \tilde{\nu}_k^0 = \partial H_0/\partial J_k^0$ we postulate for each of them an equation as in Section 14 (3):

$$w_k = w_k^0 + \lambda(\tilde{w}_1^0, \ldots \tilde{w}_f^0).$$

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[1] In Sect. 15 a direct integration was performed without transformation by a new variable. The fact that the zero vector of the coordinates is used in a different sense there is associated with this.

From this one can conclude in comparison with (2) that $S_0 = \sum J_k w_k^0$ and that all other $\partial S_n / \partial J_k$, , thus also the S_n values themselves are periodic functions of the w_k^0 values with Period One.

Taking (2), (3), (4) into consideration, after the J_k values were already introduced for J_k^0 and the members of the Hamiltonian function were duly expanded.

$$\left.\begin{aligned} H \equiv H_0(J) + \lambda\left\{\sum_k \frac{\partial H_0}{\partial J_k}\frac{\partial S_1}{\partial w_k^0} + H_1(Jw^0)\right\} \\ + \lambda^2\left\{\sum_k \frac{\partial H_0}{\partial J_k}\frac{\partial S_2}{\partial w_k^0} + \frac{1}{2!}\sum_{kj}\frac{\partial^2 H_0}{\partial J_k \partial J_j}\frac{\partial S_1}{\partial w_k^0}\frac{\partial S_1}{\partial w_j^0} + \sum_k \frac{\partial H_1}{\partial J_k}\frac{\partial S_1}{\partial w_k^0} + H_2(Jw^0)\right\} \\ + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ + \lambda^n\left\{\sum_k \frac{\partial H_0}{\partial J_k}\frac{\partial S_n}{\partial w_k^0} + \Phi_n(Jw^0) + \cdots = W.\right. \end{aligned}\right\} \quad (5)$$

$H_0(J)$, $\partial H_0/\partial J_k$ etc. means that in $H_0(J^0)$ bzw. or its derivation lie J_k^0 the J_j^0 substituted by J_j in an unchanged form of the function. The quantities $\partial H_0/\partial J_k$ are thus nothing more than the frequencies ν_k^0 of the unperturbed motion, which would assume it for the fixed values $J_k^0 = J_k$. The functions Φ_n are sums of numbers, each of which contain one of the functions $H_0 \ldots H_n$ or its derivations. And, moreover, most contain factors $\partial S_i/\partial w_k^0 (i = 1, 2, \ldots, n-1)$; thus only functions known in the n-th step, which are also all periodic in the w_k^0 values with Period 1, so that Φ_n can be written as

$$\Phi_n = \sum_{-\infty}^{+\infty} \cdots \sum_{-\infty}^{+\infty} A^{(n)}_{r_1 \ldots r_f} e^{2\pi i (r_1 w_1^0 + \cdots + r_f w_f^0)}. \quad (6)$$

From the differential equation (5) we individually draw the equations

$$\sum \nu_k^0 \frac{\partial S_n}{\partial w_k^0} + \Phi_n(Jw^0) = W_n. \quad (7)$$

S_n must be, as was shown above, a periodic function of w_k^0; thus, we state undetermined coefficients

$$S_n = \sum_{-\infty}^{+\infty} \cdots \sum_{-\infty}^{+\infty} B^{(n)}_{\tau_1 \ldots \tau_f} e^{2\pi i (\tau_1 w_1^0 + \cdots + \tau_f w_f^0)} \tag{8}$$

and gain comparison of (6), (7) and (8), when we indicate the purely periodic portion of Φ (without constant number) with $\tilde{\Phi}$

$$\sum_k \nu_k^0 \frac{\partial S_n}{\partial n_k^0} = -\tilde{\Phi}_n, \qquad B^{(n)}_{\tau_1 \ldots \tau_f} = \frac{A^{(n)}_{\tau_1 \ldots \tau_f}}{2\pi i \sum \tau_k \nu_k^0} \tag{9}$$

with the exception of $B_{00\ldots0}$, which remains arbitrary (however, is immaterial as additive constant in S). Moreover,

$$W_n = A^{(n)}_{0\ldots0} = \bar{\Phi}_n. \tag{10}$$

follows from (6) and (7). This formally completes the definition of the functions S_n, thus also those of J_k, w_k. In addition, W resulted in the function of the new action variables of form

$$W \equiv W_0(J) + \lambda W(J) + \cdots + \lambda^n W_n(J) + \cdots, \tag{11}$$

of which we give several members:

$$W_0 = H_0(J), \qquad W_1 = \overline{H_1(J)}, \qquad W_n = \overline{\Phi_n(J)}.$$

The process is best applied to the needs of quantum mechanics since it permits one to determine the n-th approximation of the energy already after n - 1 steps. The first approximation member W_1 is found to be the chronological mean value of the perturbation function of the first order taken over the unperturbed motion.

Let us observe the connection between the old and new variables

and forget for the moment the question of convergency. Then

$$\left.\begin{aligned} J_k^0 &= J_k + \sum_{n=1}^{\infty} \lambda^n \frac{\partial S_n}{\partial w_k^0}, \\ w_k &= w_k^0 + \sum_{n=1}^{\infty} \lambda^n \frac{\partial S_n}{\partial J_k}, \\ \nu_k &= \nu_k^0 + \sum_{n=1}^{\infty} \lambda^n \frac{\partial W_n}{\partial J_k}. \end{aligned}\right\} \qquad (12)$$

the unperturbed motion is overlaid by small fluctuations vanishing with λ with approximately the old finite frequency. Thus, only the so-called short periodic perturbations occur here, not the secular perturbations described in the last section.

The question of the convergence of the series contained in (8) and (9) has been discussed best in investigations by Bruns[1]. Since according to our presuppositions the frequencies ν_k^0 are incommensurable for unperturbed motion, the denominators in (9) for the old values J_k certain do not vanish exactly. In spite of this they can become as small as desired for certain combinations of the τ_k values. Bruns has shown that the numerical-theoretical character of the relationships $\nu_1^0 : \ldots : \nu_f^0$ is decisive for the convergence of divergence of the series in such a manner that infinitely many convergence and divergence positions lie in a region of the ν_k^0 values which is still so small.

As we know, the individual member of this series never converges for all rational proportions contained in it. That which holds

1) H. Bruns, Astron. Nachr., Vol. 109, p. 215, 1884; C. L. Charlier, Mechanik des Himmels, Vol. II, p. 307, cf. Sect. 1, footnote; H. Poincáre, Méthodes nouvelles, Vol. II, Ch. VIII and XIII.

true for the ν_k^0 region also holds true for the region of the J_k values on account of the uniform functional co-relationship. Thus, one comes to the conclusion that the function S defined by (8) and (9) is no uniform function of the J_k values. With this all the suppositions of the calculation are destroyed, e.g., the equations (2). In spite of this, astronomical practice shows that the series (8) are of greatest significance. Calculations of great accuracy allow that they are discontinued at suitable positions. This is due to their semi-convergence, to which Poincaré has devoted several investigations, which, however, are not final.[1]

19. Perturbations of a characteristically decomposed system. If the initial point is characteristically decomposed, the procedure from the previous section cannot be applied since the uncharacteristic angle variables are then constant in the unperturbed case, and we know from Sect. 17 that the perturbation produces secular finite changes in them. The postulate for Equation (4) from Sect. 18 and the statement (8) connected with it, as well as the completeness of the differential equation (9) are lacking for it, but are attained in that one first observes an intermediary motion which contains the secular perturbation, so that the total motion only deviates from it by the member $\infty \lambda$.

1) H. Poincaré, op. cit..

From the complete Hamiltonian function

$$\mathfrak{H} \equiv \mathfrak{H}_0(J_\alpha^0) + \lambda \mathfrak{H}_1(J_\alpha^0, J_\varrho^0, w_\alpha^0, w_\varrho^0) + \cdots + \lambda^n \mathfrak{H}_n(J_k^0, w_l^0) + \cdots = W \tag{1}$$

we next select a portion

$$\mathfrak{H}_0^* \equiv \mathfrak{H}_0(J_\alpha^0) + \lambda \mathfrak{G}(J_\alpha^0, J_\varrho^0, w_\varrho^0) = W_0 + \lambda W_1 \tag{2}$$

According to Sect. 17 we suspect that $G = \overline{\mathfrak{H}}_1$ but we delay the decision long enough so that this selection proceeds directly from the context of the following calculation. The integration of the motion problem (2) is completed with the help of the old procedure: one looks for a generatrix

$$S_0 = \sum_\alpha J_\alpha^* w_\alpha^0 + T(J_\alpha^*, J_\varrho^*, w_\varrho^0) \tag{3}$$

of the (finite) transformation

$$w_k^* = \frac{\partial S_0}{\partial J_k^*}, \qquad J_\alpha^0 = \frac{\partial S_0}{\partial w_\alpha^0} = J_\alpha^*, \qquad J_\varrho^0 = \frac{\partial S_0}{\partial w_\varrho^0} = \frac{\partial T}{\partial w_\varrho^0}$$

in new action and angle variables J_K^*, w_K^* to be determined from the differential equation (corresponding to Sect. 17, Equation (5)).

$$\mathfrak{G}\left(J_\alpha^*, \frac{\partial T}{\partial w_\varrho^0}, w_\varrho^0\right) = W_1. \tag{4}$$

A general way of solving it cannot be given (cf. the comment in the close of Sect. 14), we will assume that it is integrated. After introducing the J_K^*, W_K^* in (1)

$$H \equiv H_0^*(J_k^*) + \lambda[H_1(J_k^*, w_k^*) - G(J_k^*)] + \cdots + \lambda^n H_n(J_k^*, w_k^*) + \cdots = W. \tag{5}$$

arises (we now write the transformed forms of the function with Latin letters, however, $H_0 = \mathfrak{H}_0$). At the same time H_0^* consists

of two parts of various orders of magnetude:

$$H_0^*(J_k^*) = H_0(J_\alpha^*) + \lambda G(J_\alpha^*, J_\varrho^*).$$

This also causes the motion frequencies ν_k^* of the previously corresponding and non-corresponding angle variables to have various orders of magnitude:

$$\nu_\alpha^* = \frac{\partial H_0^*}{\partial J_\alpha^*} \sim 1, \qquad \nu_\varrho^* = \frac{\partial H_0^*}{\partial J_\varrho^*} = \lambda \frac{\partial G}{\partial J_\varrho^*} \sim \lambda. \tag{6}$$

If one now wanted to apply the procedure from Section 18 to (5), i.e., introduce the final variable J_k, w_k with the help of

$$\left.\begin{aligned} S &= \sum_k J_k w_k^* + \sum_{n=1}^{\infty} \lambda^n S_n(J_k, \tilde{w}_k^*), \\ w_k &= w_k^* + \sum_n \lambda^n \frac{\partial S_n}{\partial J_k}; \qquad J_k^* = J_k + \sum_n \lambda^n \frac{\partial S_n}{\partial w_k^*} \end{aligned}\right\} \tag{7}$$

as a result of (6) of coefficients $B^{(n)}_{\tau_1 \ldots \tau_f}$, for which the τ_α values are equal to 0 would be proportional $1/\lambda$, because of the denominator in Sect. 18 (9). I.e., a certain part which already belonged to the (n-1)th approximation would be extracted.

In the case of the first approximation this can be prevented by causing all portions of the member from (5) subject to λ which are independent of the w_α^* values, i.e., all amplitudes $A^{(n)}_{\tau_1 \ldots \tau_f}$ $(\tau_\alpha = 0)$ to vanish. This occurs by the selection of $G = \bar{H}_1$, which is found to be free of arbitrariness (one subsequently also sees that it was permitted to assume that $\mathfrak{S}$ was independent of the w_α^0 values from the beginning. (The secular perturbations are completely defined

by the function $\overline{\mathfrak{H}}_1(w_e^o)$, and the calculations from Sect. 17 are found to be beginning of a detailed approximation procedure. In the later approximations, however, one cannot avoid the determination of each function S_{n-1} in two steps[1]. One, therefore, expands the function (5) with the help of (7) on the position $J_k^* = J_k$ and if one considers that $G = \bar{H}_1$, one then obtains:

$$H_0(J_\alpha) + \lambda\left\{\sum_\alpha \frac{\partial H_0}{\partial J_\alpha}\frac{\partial S_1}{\partial w_\alpha^*} + \bar{H}_1(J_k, w_k^*) + \bar{H}_1(J_k)\right\}$$
$$+\sum_{n=2}^{\infty}\lambda^n\left\{\sum_\alpha \frac{\partial H_0}{\partial J_\alpha}\frac{\partial S_n}{\partial w_\alpha^*} + \sum_k \frac{\partial \bar{H}_1}{\partial J_k}\frac{\partial S_{n-1}}{\partial w_k^*} + \Phi_n(S_1, \ldots S_{n-1}, H_0, \ldots H_n)\right\} = W.$$

The equation corresponding to equation (7) from Sect. 18

$$\sum_\alpha \frac{\partial H_0}{\partial J_\alpha}\frac{\partial S_n}{\partial w_\alpha^*} + \sum_k \frac{\partial \bar{H}_1}{\partial J_k}\frac{\partial S_{n-1}}{\partial w_k^*} + \Phi_n(S_1, \ldots S_{n-1}, H_0, \ldots H_n) = W_n \quad (8)$$

can primarily only be averaged over the unit cube of all w_k^*, which we will indicate with two bars, and produces, since neither of the two sums possess a constant member,

$$W_n = \bar{\Phi}_n.$$

One notices that nothing more is added to the initially calculated values of $W_1 (= \bar{H}_1)$ and thus the following theorem also holds true here: the first correction of the energy value is equal to the mean time value of the perturbation energy of the first order, taken over the unperturbed path. Second, one can average it over the unit cube of the W_α^* values alone (i.e., over the chronological course of the unperturbed path), which

1. Opposed to this, an opinion earlier expressed by the author (2 S.f. Phys. Bd. 35, p. 224. 1925).

was previously indicated by one bar. The result

$$\sum_{\varrho} \frac{\partial \bar{H}_1}{\partial J_\varrho} \frac{\partial S_{n-1}}{\partial w_\varrho^*} + \bar{\Phi}_n = W_n \qquad (9)$$

was taken from (8). The difference

$$\sum_{\alpha} \frac{\partial H_0}{\partial J_\alpha} \frac{\partial S_n}{\partial w_\alpha^*} + \sum_{\alpha} \frac{\partial \bar{H}_1}{\partial J_\alpha} \frac{\partial S_{n-1}}{\partial w_\alpha^*} + \tilde{\Phi}_n(S_1, \ldots S_{n-1}, H_0, \ldots H_n) = 0.$$

arises from this. The portion S_n' from S_n is determined as previously from this linear differential equation through proficient comparison of the Fourier series. A further portion R_n, which is only dependent on the variables w_ϱ^*, remains undefined. It can subsequently be determined in a similarly simple fashion, however, from the equation

$$\sum_{\varrho} \frac{\partial H_1}{\partial J_\varrho} \frac{\partial R_n}{\partial w_\varrho^*} = W_{n+1} - \bar{\Phi}_{n+1} = -\tilde{\Phi}_{n+1}$$

corresponding to (9) since it has been determined that $\bar{\Phi}_{n+1}$ does indeed depend on S_n', but not on R_n, so that the known Fourier numbers appear again on the right.

The procedure is not valid for the practically important case of $\bar{H}_1 = 0$. In this case one first has to eliminate H_1 by a transformation of form (7) and has to calculate the secular perturbations from the mean value of the new function H_2 which then results. Thus, not only the mean value of the original function H_2 lends reliability to them, but also a member stemming from the short periodic perturbations of the first order[1].

1. One can find more on this in a work by M. Born and W. Heisenberg, Ann. d. Phys., Vol. 74, p. 1, 1924.

20. Perturbation of a randomly decomposed system. The method one must use in order to investigate the perturbations of a conditionally periodic system in the neighborhood of a point of random decomposition is of a somewhat different form for the following reason: in principle one must always maintain that the action variables of the perturbed system are indeed natural constants of the motion but are variable quantities from path to path. It is only in this manner that the generatrics $S(J, w^0)$ of the transformation $(J^0; w^0) \rightarrow (J, w)$ can be considered a function of the same and can attribute a meaning to the transformation equations

$$J_k^0 = \frac{\partial S}{\partial w_k^0}, \qquad w_k = \frac{\partial S}{\partial J_k} \tag{1}$$

Physically, the need also arises to investigate not only the perturbation of the one, exact randomly decomposed motion, but also its neighboring motion (for which the critical frequencies do not, in fact, vanish, but become very small). Mathematically, the position of random decomposition is characterized by quite definite values $J_\alpha^0 = J_\alpha^*$, $J_\varepsilon^0 = J_\varepsilon^*$ of the characteristics and non-characteristic action variables of the initial system. The peculiarity of this case appears most clearly when one holds completely to them in calculation and makes certain, for example, that the new expansion of the Hamiltonian function (cf. Equation (5) was acquired in Sect. 18) is undertaken at the critical point

itself. Thus, things must be arranged so that the left series of the transformation equations (1) contains the form

$$J_k^o = J_k^* + \text{members which vanish with } \lambda \ ;$$

thus, S is to be written

$$S = \sum J_k^* w_k^o + \text{a segment } S', \text{ which vanishes with } \lambda \ . \quad (2)$$

This is the one and probably most important difference from the earlier cases. There is a second. While during characteristic decomposition of a system the decomposed angle and action variables carry with them a certain arbitrariness (they are, in fact, coordinates for degrees of freedom, which are not used in any motion of the system; they are therefore not characterized by the motion either), this arbitrariness in the selection of the non-characteristic variables is only eliminated during random decomposition. Here they are uniquely defined by the possible neighboring motions. This is the inner reason for the fact that the secular perturbations, which again must first be investigated, are not applied as strongly here, so that an approximation procedure is sufficient to calculate them.

After these preliminary remarks we turn to the task itself, whose solution stems from Bohlin[1].

The Hamiltonian function can be written in the action and

1. H. Bohlin, Behang till K. Svenska Vet. Akad. Handlinger, Vol. 14, Issue I, No. 5. Stockholm 1888; cf. H. Poincare, Méthodes nouvelles Vol. II, Ch XIX and XX; M. Born and W. Heisenberg, ZS. f. Phys. Vol 14, p. 44. 1923.

angle variables of the unperturbed system

$$H \equiv H_0(J_\alpha^0, J_\varrho^0) + \lambda H_2(J_\alpha^0, J_\varrho^0, w_\alpha^0, w_\varrho^0) + \cdots + \lambda^n H_{2n}(J^0, w^0) + \cdots = W. \tag{3}$$

(The indication of the H_i values only with even numbers will become understandable later.) H_0 thus also depends on the J^0 values, but with the conditions

$$\nu_\varrho^* = \frac{\partial H_0}{\partial J_\varrho^*} = 0; \tag{4}$$

$\partial H_0 / \partial J_\varrho^*$ means, as earlier, $\partial H_0 / \partial J_\varrho^0$ $(J_\alpha^*, J_\varrho^*)$.

The problem is primarily to calculate the secular perturbations and, thus, to acquire a non-decomposed intermediary motion. They are obtained, as in Sect. 17 and 19, from

$$H_0^* \equiv H_0(J_\alpha^0, J_\varrho^0) + \lambda \bar{H}_2(J_\alpha^0, J_\varrho^0, w_\varrho^0) = W_0^*, \tag{5}$$

a problem of only s degrees of freedom, in which the J_α^0 values play the role of constant parameters. In many cases it is possible and advisable to integrate the same strickly; in other cases the Bohlinian approximation procedure, which we will now turn to, is to be applied. If one then defines the generatrics T for the transformation on the intermediary angle and action variables $w_k, \Im_k$ of (5) from the Hamiltonian differential equation

$$H_0^* = H_0\left(\frac{\partial T}{\partial w_\alpha^0}, \frac{\partial T}{\partial w_\varrho^0}\right) + \lambda \bar{H}_2\left(\frac{\partial T}{\partial w_\alpha^0}, \frac{\partial T}{\partial w_\varrho^0}, w_\varrho^0\right) = W_0^*, \tag{6}$$

then the statement can and should, corresponding to (2) for T

$$T(\Im_k, w_k^0) = \sum J_k^* w_k^0 + T'(\Im_k, w_k^0),$$

wherein T' vanishes with λ . Moreover, (6) can be separated in $\partial T/\partial w_\alpha^0 = \Im_\alpha$ = constant and the corresponding remaining equation. In the transformation equations

$$J_k^0 = \frac{\partial T}{\partial w_k^0} = J_k^* + \frac{\partial T'(\Im_k w_k^0)}{\partial w_k^0} \tag{7}$$

one can thus assume

$$\frac{\partial T'}{\partial w_k^0} = \Delta_k$$

to be constant [namely, equal to $(\Im_\alpha - J_\alpha^*)$] for the non-decomposed degrees of freedom.

Bohlin then showed that the expansion of T' must proceed by powers of $\sqrt{\lambda}$. This can perhaps be understood by the following train of thought: If one introduces (7) into (5) then

$$W_0^* - H_0(J_\alpha^*, J_\varrho^*) - \sum_\alpha \nu_\alpha^* \Delta_\alpha = \left\{\sum_\varrho \sum_\sigma A_{\varrho\sigma}\Delta_\varrho\Delta_\sigma + \sum_\varrho B_\varrho \Delta_\varrho + C\right\} + \cdots + \lambda \bar{H}_2$$

with the definitions

$$A_{\varrho\sigma} = \frac{1}{2!}\frac{\partial^2 H_0}{\partial J_\varrho^* \partial J_\sigma^*}, \qquad B_\varrho = \sum_\alpha \frac{1}{2!}\frac{\partial^2 H}{\partial J_\varrho^* \partial J_\alpha^*}\Delta_\alpha,$$

$$C = \sum_\alpha \sum_\beta \frac{1}{2!}\frac{\partial^2 H_0}{\partial J_\alpha^* \partial J_\beta^*}\Delta_\alpha \Delta_\beta .$$

The left side is constant since the quantities Δ_α are arbitrary constants which only serve the purpose of freeing the $\Im_\alpha$ from the fixed values J_α^* . One sees that the variable quantities Δ_ϱ must be of order $\sqrt{\lambda}$ in order to equally divide out the changes in $\bar{H}_2$. We thus postulate that

$$T = \sum J_k^* w_k^0 + \sqrt{\lambda}\, T_1 + \lambda T_2 + \cdots \tag{8}$$

and obtain from (6) with $a_{\varrho\sigma} = A_{\varrho\sigma}$, $\sqrt{\lambda}\, b_\varrho = B_\varrho$, $\lambda c = C$

$$\left.\begin{aligned} &H_0(J_\alpha^*, J_\varrho^*) + \sqrt{\lambda}\sum_\alpha \nu_\alpha^* \frac{\partial T_1}{\partial w_\alpha^0} \\ &\quad + \lambda\left\{\sum_\varrho\sum_\sigma a_{\varrho\sigma}\frac{\partial T_1}{\partial w_\varrho^0}\frac{\partial T_1}{\partial w_\sigma^0} + \sum_\varrho b_\varrho \frac{\partial T_1}{\partial w_\varrho^0} + c + \bar{H}_2(J_\alpha^*, J_\varrho^*, w_\varrho^0)\right\} \\ &\quad + \lambda\sqrt{\lambda}\{\}_3 + \lambda^2\{\}_4 + \cdots = W_0^* \equiv W_0 + \sqrt{\lambda}\, W_1 + \lambda W_2 + \cdots \end{aligned}\right\} \tag{6'}$$

and from this partial equation

$$H_0(J_\alpha^*, J_\varrho^*) = W_0, \tag{9_0}$$

$$\sum_\alpha \nu_\alpha^* \frac{\partial T_1}{\partial w_\alpha^0} = W_1, \tag{9_1}$$

$$\sum_\varrho \sum_\sigma a_{\varrho\sigma} \frac{\partial T_1}{\partial w_\varrho^0} \frac{\partial T_1}{\partial w_\sigma^0} + \sum_\varrho b_\varrho \frac{\partial T_1}{\partial w_\varrho^0} + c + \overline{H}_2(J_\alpha^*, J_\varrho^*, w_\varrho^0) = W_2. \tag{9_2}$$

the constancy of $\partial T_1/\partial w_\alpha^0$, which was to be predicted, follows from (9_1); we therefore postulate, by including in T_1 primarily those integration constants $\mathfrak{K}$ which happen to appear directly, and only later substitute them by the action variables $\mathfrak{J}$,

$$\frac{\partial T_1}{\partial w_\alpha^0} = \mathfrak{K}_\alpha$$

and obtain

$$T_1 = \sum_\alpha \mathfrak{K}_\alpha w_\alpha^0 + T_1'(w_\varrho^0).$$

T_1' is defined from (9_2). In spite of the fact that this differential equation has a much more simple character than (6) which it represents, no general solution for any given H_2 can be given. The Bohlinian approximation procedure does indeed simplify the calculation of the secular perturbations (this is all it does), but it does not make them mandatory. Nevertheless, we assume that (9_2) is integrable in some way or other, say, by separation. In the result one obtains T_1 as a function of the w_ϱ^0 and from s arbitrary integration constants, in whose place we introduce the phase integral

$$\mathfrak{K}_\varrho = \oint \frac{\partial T_1}{\partial w_\varrho^0} dw_\varrho^0$$

as in Section 6 so that

$$T_1 = \sum_\alpha \mathfrak{K}_\alpha w_\alpha^0 + T_1'(\mathfrak{K}_\varrho, w_\varrho^0) \tag{8'}$$

arises.

Here we end the approximation procedure. It only serves the purpose of providing the secular perturbations in the first approximation. We therefore arbitrarily set $T_2 = T_3 = \ldots = 0$, but have not integrated (6) or (6'), but the problem

$$H_0^* - \lambda\sqrt{\lambda}\{\}_3 - \lambda^2\{\}_4 - \cdots = \text{illegible} + \sqrt{\lambda}W_1 + \lambda W_2. \tag{10}$$

A further comment must be made on the introduced quantities $\mathfrak{K}_k$. They are not the action variables of (10), which one already sees, since they are not transformed into the values J_k^* for $\lambda = 0$. These would be defined according to Section 6 (1) by phase integrals

$$\mathfrak{J}_k = \oint \frac{\partial T}{\partial w_k^0} dw_k^0.$$

According to (8) and (8') this produces

$$\left.\begin{aligned} \mathfrak{J}_\alpha &= J_\alpha^0 = J_\alpha^* + \sqrt{\lambda}\mathfrak{K}_\alpha, \\ \mathfrak{J}_\varrho &= J_\varrho^* \oint dw_\varrho^0 + \sqrt{\lambda}\mathfrak{K}_\varrho, \end{aligned}\right\} \tag{11}$$

and the canonically conjugated angle variables with respect to H are

$$\begin{aligned} \mathfrak{w}_\alpha &= \frac{\partial T}{\partial \mathfrak{J}_\alpha} = \frac{\partial T_1}{\partial \mathfrak{K}_\alpha} = w_\alpha^0, \\ \mathfrak{w}_\varrho &= \frac{\partial T}{\partial \mathfrak{J}_\varrho} = \frac{\partial T_1}{\partial \mathfrak{K}_\varrho}. \end{aligned}$$

It is more useful for what follows to retain the variable pair $\mathfrak{w}_k, \mathfrak{K}_k$ canonically conjugated with respect to $\frac{1}{\sqrt{\lambda}}(H - W_0)$. Thus, a canonical transformation of the character of Equation (11) in Chapter 3, Section 3 is woven into the calculation.

One must remember, however, that the quantities $\mathfrak{A}$ do not possess the property of being action variables.

In the coordinates selected the complete problem (3) to which we now turn is written as

$$\frac{H-W_0}{\sqrt{\lambda}} = W_1(\mathfrak{K}_\alpha) + \sqrt{\lambda}\{W_2(\mathfrak{K}_\alpha, \mathfrak{K}_\varrho) + \tilde{H}_2(\mathfrak{K}, \mathfrak{w})\} + \lambda\{\mathfrak{K}, \mathfrak{w}\}_3 + \cdots = \frac{W-W_0}{\sqrt{\lambda}}$$

or

$$\mathfrak{H} \equiv \mathfrak{H}_0(\mathfrak{K}_\alpha) + \sqrt{\lambda}\{\mathfrak{H}_1(\mathfrak{K}_\varkappa, \mathfrak{K}_\varrho) + \tilde{\mathfrak{H}}_1(\mathfrak{K}, \mathfrak{w})\} + \cdots + \sqrt{\lambda}^n \mathfrak{H}_n(\mathfrak{K}, \mathfrak{w}) + \cdots = W,$$

$$i\mathfrak{w}_k = \frac{\partial \mathfrak{H}}{\partial \mathfrak{K}_k}, \qquad \dot{\mathfrak{K}}_k = -\frac{\partial \mathfrak{H}}{\partial \mathfrak{w}_k}.$$

One sees that we have arrived at a characteristically decomposed problem whose secular perturbations are already known, and that the perturbation calculation from Section 19 is admissible. This leads to new variables w_k, K_k, (the former are the final angle variables, while the K_k are only canonically conjugated with respect to $\mathfrak{H}$. In order to finally arrive at the action variable J_k of the perturbed motion which are conjugated with respect to H, one must make use of a reverse transformation of the variety cited in Chapter 3, Section 3, Equation (11). However, this offers no difficulty.

The case where $\mathfrak{K}_\alpha = \mathfrak{K}_\varrho = 0$, is alone important for quantum theory since the equation (11) would produce a continual dependency of the $\mathfrak{J}_k$ on λ for every other value of the $\mathfrak{A}_\kappa$, which contradicts the basic premeses of this theory (principle of adiabatic in variance, cf. Section 11). With $\mathfrak{K}_\alpha = 0$ it follows that $b_\varrho = c = 0$.

and Equation (9_2) assumes the simple form

$$\sum_{\varrho}\sum_{\sigma} a_{\varrho\sigma}\frac{\partial T_1}{\partial w_{\varrho}^0}\frac{\partial T_1}{\partial w_{\sigma}^0} + \overline{H}_2(J_{\alpha}^*, J_{\varrho}^*, w_{\varrho}^0) = W_2,$$

of a motion corresponding to the Hamiltonian function

$$h = \sum_{\varrho}\sum_{\sigma} a_{\varrho\sigma} p_{\varrho} p_{\sigma} + \overline{H}_2(J_{\alpha}^*, J_{\varrho}^*, q_{\varrho})$$

one sees that the canonical equation

$$\dot{q}_{\varrho} = \frac{\partial h}{\partial p_{\varrho}}, \qquad \dot{p}_{\varrho} = -\frac{\partial h}{\partial q_{\varrho}}$$

are made sufficient by

$$p_{\varrho} = \frac{\partial T_1}{\partial w_{\varrho}^0} = 0, \qquad q_{\varrho}\left(\text{equals the roots of } \frac{\partial \overline{H}_2}{\partial q} = 0\right) = \text{constant}$$

This is the motion with $\mathfrak{R}_{\kappa}=0$. It seems as if the important quantum-theoretical special solution in which the perturbed motion possesses the same degree of decomposition as the unperturbed, would be possible for every randomly decomposed system and with any given forces of perturbation. One must consider, however, that our conclusions only applied to the first approximation, and in higher approximations the existence of this solution is still questionable (cf. Section 22).

21. Example of the perturbation of a limit-decomposed system. The important case in quantum theory described in the previous section in which the decomposed variables remain about the center of libration, cannot be developed in the manner described for higher approximations. The method fails for the simple reason that the

functions $\mathfrak{H}_n$ permit no Taylor expansion at the critical points $\mathfrak{J}_k = J_k^*, \mathfrak{K}_k = 0$

A simple example may clarify the development of this difficulty and also show that it is not limited to the case described.

We observe the motion of a linear harmonic oscilator whose direction of vibrations is vertical under the perturbing influence of the force of gravity, which, however, can be thought of as small with respect to the quasi-elastic bonding in the rest state. The Hamiltonian function of this system in the usual coordinates (cf. Section 5, conclusion) is

$$H \equiv \frac{1}{2m} p_z^2 + 2\pi^2 \nu^2 m z^2 + \lambda z = W. \qquad (1)$$

After the introduction of the angle and action variables the unperturbed system with the help of a Poincaré transformation

cf. Section 5, Equation (8)

$$z = \sqrt{\frac{J^0}{2\pi^2 \nu m}} \sin 2\pi w^0, \qquad p_z = \sqrt{2\nu m J^0} \cos 2\pi w^0 \qquad (2)$$

The form

$$H \equiv \nu J^0 + \lambda \sqrt{\frac{J^0}{2\pi^2 \nu m}} \sin 2\pi w^0 \equiv H_0 + \lambda H_1 = W. \qquad (3)$$

develops. As in Section 18, Equation (12) One postulates

$$J^0 = J + \lambda \cdot (\tilde{w}^0) + \cdots$$

and again expands H by powers of λ . One then sees that the old procedure leads to the calculation of the perturbed motion for every value of J except for J=0, i.e., when z remains in the center of libration. Then numerous derivations of the perturbation energy H_1 become infinite. (more exactly, it is an expansion by powers of λ is no longer possible, it is only possible by $\lambda / \sqrt{J}$, which allows one to recognize the region of convergence.)

The strict solution of the problem, which was easily found in this case and consists, as is known, of a harmonic oscilation about a displaced center, shows that the difficulty is only of a formal nature.

Apparently it is associated with the introduction of unsuitable coordinates. It was explained in Section 8 why the representation in angle and action variables is unsuitable for vanishing librations. Actually, calculation with librating coordinates z, p_z would have presented no difficulty. Moreover, the position of the displaced center of libration as well as the inertia in it could be recognized as the possible state of motion from (1) and the canonical equations $\dot{z} = \frac{\partial H}{\partial p_z} = 0, \quad \dot{p}_z = -\frac{\partial H}{\partial z} = 0$

22. Perturbation for limit-decomposition in the general case. The difficulty discussed in Section 21 always appears when the libration of any given coordinate of the unperturbed system occurs with vanishing amplitude; cf. Section 8. (Moreover, viewed from a purely mathematical standpoint it can also appear in other cases, namely always when one of the function H_n permits no Taylor expansion for the values J_k considered). In the direct vicinity of the libration center the binding is always quasi-elastic since the expansion of the potential begins there with the square of the dying out vibrations. Every additional perturbation potential will increase,

however, in general (as the gravity potential in the example from Section 21) even with the uneven powers of the dying out vibrations. For this reason the governing action variable in the amplitudes of the perturbation function always appears under square roots for the degree of freedom to be considered. The understanding gained from our example can be almost literally transferred to the general case. Since the representation in angle and action variables is unsuitable here, for limit decomposed degrees of freedom, one departs from it and through a reversed Poincaré transformation

(Generatrix $S = \frac{\eta^{0^2}}{2} \operatorname{tg} 2\pi w^0$)

$$\xi_\varrho^0 = \sqrt{\frac{J_\varrho^0}{\pi}} \sin 2\pi w_\varrho^0, \qquad \eta_\varrho^0 = \sqrt{\frac{J_\varrho^0}{\pi}} \cos 2\pi w_\varrho^0 \tag{1}$$

turns to the coordinates $\xi_\varrho^0, \eta_\varrho^0,$ which assume librating values for small occilations (in the example it was z and the corresponding impulse p_z). Written in them, all perturbation functions expand according to whole powers, also in the center of libration, so that the Hamiltonian function assumes the form

$$H \equiv H_0 + \lambda H_1 + \cdots + \lambda^n H_n + \cdots = W,$$

wherein

$$H_0 = H_{00}(J_\alpha^0) + \sum_\varrho \sum_\sigma (c_{0\varrho\sigma}\xi_\varrho^0\xi_\sigma^0 + d_{0\varrho\sigma}\xi_\varrho^0\eta_\sigma^0 + e_{0\varrho\sigma}\eta_\varrho^0\eta_\varrho^0) + \cdots,$$

$$H_n = H_{n0}(J_\alpha^0, \tilde{w}_\alpha^0) + \sum_\varrho (a_{n\varrho}\xi_\varrho^0 + b_{n\varrho}\eta_\varrho^0) + \sum_\varrho \sum_\sigma (c_{n\varrho\sigma}\xi_\varrho^0\xi_\sigma^0 + d_{n\varrho\sigma}\xi_\varrho^0\eta_\sigma^0 + e_{n\varrho\sigma}\eta_\varrho^0\eta_\sigma^0) + \cdots.$$

Thereby, $a_{n\varrho}, b_{n\varrho}, c_{n\varrho\sigma}, \cdots$ are periodic functions of the w_α^0 values in whose amplitudes the J_α^0 also take effect. The former occurs with the

of the case n = 0 since the quantities $c_{0\varrho\sigma}, d_{0\varrho\sigma}, \ldots$ can only be dependent on the J_α^0 values; otherwise we may not view the unperturbed system as generally integrated. H_{no} signifies that function which develops from H_n with $\xi_\varrho^0 = \eta_\varrho^0 = 0$.

Those motions of the perturbed system for which the limit decompostion continues to exist or for which only small oscillations occur around the center of libration is of particular interest in astronomy, as well as atomic mechanics. One must consider though that the latter experiences a displacement by the addition of the perturbation potentials. One thus introduces the new coordinates $\xi_\varrho = \xi_\varrho^0 - A_\varrho, \quad \eta_\varrho = \eta_\varrho^0 - B_\varrho$

whose initial point lies in the center of libration. One must thereby recognize that the function H written in the new coordinates is free in all numbers of the first powers of the quantities $\xi_\varrho, \eta_\varrho$. The step by step determination of the quantities A_ϱ and B_ϱ according to this point of view can be linked with the successive introduction of the new angle variables for the non-decomposed degrees of freedom. The generatrix of the transformation can then be written generally as

$$S = \sum_\alpha J_\alpha w_\alpha^0 + T(J_\alpha, \tilde{w}_\alpha^0) + \sum_\varrho (\xi_\varrho^0 \eta_\varrho + B_\varrho \xi_\varrho^0 - A_\varrho \eta_\varrho),$$

whereby the quantities T, A_ϱ , B_ϱ are power series in λ and periodic

with the w^0_α values:

$$\begin{aligned} T &= \lambda T_1 + \lambda^2 T_2 + \cdots, \\ A_\varrho &= \lambda A_{1\varrho} + \lambda^2 A_{2\varrho} + \cdots, \\ B_\varrho &= \lambda B_{1\varrho} + \lambda^2 B_{2\varrho} + \cdots. \end{aligned}$$

with the relationships following from this

$$\begin{aligned} J^0_\alpha &= J_\alpha + \lambda\left(\frac{\partial T_1}{\partial w^0_\alpha} + \sum_\varrho \xi_\varrho \frac{\partial B_{1\varrho}}{\partial w^0_\alpha} - \sum_\varrho \eta_\varrho \frac{\partial A_{1\varrho}}{\partial w^0_\alpha}\right) \\ &\quad + \lambda^2\left(\frac{\partial T_2}{\partial w^0_\alpha} + \sum_\varrho \xi_\varrho \frac{\partial B_{2\varrho}}{\partial w^0_\alpha} - \sum_\varrho \eta_\varrho \frac{\partial A_{2\varrho}}{\partial w^0_\alpha} + \sum_\varrho A_{1\varrho} \frac{\partial B_{1\varrho}}{\partial w^0_\alpha}\right) + \cdots, \\ \xi^0_\varrho &= \xi_\varrho + \lambda A_{1\varrho} + \lambda^2 A_{2\varrho} + \cdots, \\ \eta^0_\varrho &= \eta_\varrho + \lambda B_{1\varrho} + \lambda^2 B_{2\varrho} + \cdots \end{aligned}$$

a new expansion of H occurs, whose beginning is reproduced here at least for the case of a limit-decomposed degree of freedom:

$$\begin{aligned} H &\equiv H_{00}(J_\alpha) + (c_0\xi^2 + d_0\xi\eta + e_0\eta^2) + \text{ higher powers in } \xi, \eta \\ &+ \lambda\Big\{\sum_\alpha \frac{\partial H_{00}}{\partial J_\alpha}\frac{\partial T_1}{\partial w^0_\alpha} + \xi\sum_\alpha \frac{\partial H_{00}}{\partial J_\alpha}\frac{\partial B_1}{\partial w^0_\alpha} - \eta\sum_\alpha \frac{\partial H_{00}}{\partial J_\alpha}\frac{\partial A_1}{\partial w^0_\alpha} \\ &+ \xi(2c_0A_1 + d_0B_1) + \eta(d_0A_1 + 2e_0B_1) + H_{10}(J_\alpha, w^0_\alpha) \\ &\qquad + a_1\xi + b_1\eta + \text{ quadratic and higher powers in } \xi, \eta\Big\} \\ &+ \lambda^2\{\} + \cdots = W. \end{aligned}$$

From this the definition equation follows for the first approximation

$$H_{00}(J_\alpha) + R_0(\xi, \eta, J_\alpha) = W_0 \tag{2}$$

with $$R_0 = c_0\xi^2 + d_0\xi\eta + e_0\eta^2 + \cdots$$

In the second approximation

$$\sum_\alpha \frac{\partial H_{00}}{\partial J_\alpha}\frac{\partial T_1}{\partial w^0_\alpha} + H_{10} + R_1(\xi, \eta, J_\alpha, w^0_\alpha) = W_1, \tag{3}$$

with the additional conditions that the factors of ξ and η must

vanish

$$\sum_{\alpha} \frac{\partial H_{00}}{\partial J_\alpha} \frac{\partial B_1}{\partial w_\alpha^0} + 2c_0 A_1 + d_0 B_1 + a_1 = 0, \qquad (3')$$

$$-\sum_{\alpha} \frac{\partial H_{00}}{\partial J_\alpha} \frac{\partial A_1}{\partial w_\alpha^0} + d_0 A_1 + 2e_0 B_1 + b_1 = 0, \qquad (3'')$$

so that

$$R_1 = \sum_{\alpha} \frac{\partial T_1}{\partial w_\alpha^0} \left(\frac{\partial c_0}{\partial J_\alpha} \xi^2 + \frac{\partial d_0}{\partial J_\alpha} \xi\eta + \frac{\partial e_0}{\partial J_\alpha} \eta^2 \right) + \cdots$$
$$+ c_1 \xi^2 + d_1 \xi\eta + e_1 \eta^2 + \cdots \equiv C_1 \xi^2 + D_1 \xi\eta + E_1 \eta^2 + \cdots$$

remains for R_1. The equations (3') (3") define A_1, B_1 as Fourier series of the w_α^0; they are not significantly different from the definition equations otherwise common in perturbation theory. The situation is otherwise with Equation (3) for T_1. If one could assume that in R_1 (which only contains powers of ξ, η beginning with the second) the w_α^0 only appear in the same combinations as in the rest of the equation (3) i.e., that the equation would be separable; then, after dividing the constants W_1 into two other $U_1 + V_1$, the definition equation for T_1 would result.

$$\sum_{\alpha} \frac{\partial H_{00}}{\partial J_\alpha} \frac{\partial T_1}{\partial w_\alpha^0} + H_{10} = U_1(J_\alpha).$$

From this, T_1 would follow in the usual manner

$$R_1(U_1, \xi, \eta) = W_1 - U_1$$

and would remain as a function of J_α, ξ and η alone.

Apparently this prerequisite, which expresses the conditionally periodic character of the perturbed system, is not fulfilled in general; and a method is lacking at this point which formally produces this characteristic--at the cost of convergence. This

deficiency has not been considered in many studies on the subject. We limit ourselves here (as in the calculation of secular perturbations simply to the case in which equation (3) is separable and the procedure is continuable. Three new functions A_n, B_n, T_n for which three definition equations are available appear in every step (for the definition of T_n one must note, however, each time the assumption of reiterated separability). Thus, when one still substitutes the constants H_{oo} by $U_o(J_\alpha)$ and the coefficients c_o, d_o, e_o in (2) by the symbols C_o, D_o, E_o a gradual transformation of the Hamiltonian function occurs in the form

$$\left.\begin{aligned} H &\equiv U_0(J_\alpha) + R_0(\xi, \eta, J_\alpha) \\ &+ \lambda\{U_1(J_\alpha) + R_1(\xi, \eta, J_\alpha)\} + \cdots \\ &+ \lambda^n\{U_n(J_\alpha) + R_n(\xi, \eta, J_\alpha)\} + \cdots = W \end{aligned}\right\} \tag{4}$$

or with

$$U = \sum_{n=0}^{\infty} \lambda^n U_n, \qquad C = \sum_{n=0}^{\infty} \lambda^n C_n, \qquad D = \sum_{n=0}^{\infty} \lambda^n D_n \quad \text{usf.},$$

$$H \equiv W_0(J_\alpha) + U(J_\alpha) + C(J_\alpha)\xi^2 + D(J_\alpha)\xi\eta + E(J_\alpha) + \cdots = W.$$

For the segment dependent on ξ, η with W - U = V = constant.

$$R \equiv C\xi^2 + D\xi\eta + E\eta^2 + \cdots = V. \tag{5}$$

For small values of ξ, η (not considering the higher powers as the second) small vibrations around the center of libration result when the expressed quadratic member is definite, i.e. when after a suitable linear transformation of the ξ, η into Ξ, H and of R into

$$A\Xi^2 + BH^2 + \cdots = V \tag{6}$$

A and B have the same sign.

The calculation was only indicated for the case of a limit-decomposed degree of freedom. Initially, these factors are given,

however, by the formulations for s-fold decompostion. After the completion of the proper calculation an expression like (4) results when the limit-decomposed degrees of freedom can be separated from the others. The equation (5) which followed from this consists of a sum over all degrees of freedom ϱ. By the proper linear transformation of $\xi_\varrho, \eta_\varrho$ it can be transformed into an expression like (6)

$$R \equiv \sum_\varrho (A_\varrho \Xi_\varrho^2 + B_\varrho H_\varrho^2 + \cdots) = V,$$

which permits separation without considering the members of higher degrees. Thus, there are no new difficulties for motions in the nearest vicinity to the (multiple) center of libration. A logical transposition of the previous follows from the oscilation character of the solution.

23. A special solution for the perturbed limit-decomposed system. In the conclusion of Section 22 Equation (3) was used to show that in the general case no separation of the w_α^0 values from the $\xi_\varrho, \eta_\varrho$ values will be possible and that the method will not therefore lead to the calculation of smaller oscilations around the center of libration. However, in every case it is possible to show that $\xi_\varrho = 0, \eta_\varrho = 0$, i.e., inertia in the center of libration, is a possible motion, since the equations (3') and (3'') from Section 22 can be fulfilled by the proper selection of A_1, B_1, and since, in any event, H takes on the form

$$H \equiv H^*(J_\alpha, w_\alpha^0) + R(J_\alpha, w_\alpha^0, \xi_\varrho, \eta_\varrho) = W,$$

wherein R is a power series in $\xi_\varrho, \eta_\varrho$ beginning with members of the

second degree. The canonical equations

$$\dot{\xi}_\varrho = \frac{\partial H}{\partial \eta_\varrho}, \qquad \dot{\eta}_\varrho = -\frac{\partial H}{\partial \xi_\varrho}$$

are thus certainly fulfilled by

$$\xi_\varrho = \eta_\varrho = 0, \qquad \dot{\xi}_\varrho = \dot{\eta}_\varrho = 0.$$

However, this stabilization in the center of libration represents no standstill of the $\xi^0_\varrho, \eta^0_\varrho$ values used earlier. Moreover,

$$\xi^0_\varrho = A_\varrho(\tilde{w}^0_\alpha), \qquad \eta^0_\varrho = B_\varrho(\tilde{\omega}^0_\alpha),$$

and this motion of the $\xi^0_\varrho, \eta^0_\varrho$ continues to the definition of the phase integral J_α. Its contribution is, however, as can easily calculate, from the order λ^2, and for this reason the equation (3) from Section 22 for the case $\xi = \eta = R_1 = 0$ permits the definition of T_1 without considering the center of libration of the decomposed degrees of freedom.

The special case of the motion considered in this section is the only important case from the standpoint of quantum theory. It possesses the same degree of decomposition as the unperturbed motion and, in the first approximation, also the same influence of the non-decomposed degrees of freedom through the perturbation forces, as if the decomposed variables were not being acted upon. In the second approximation, however, the interaction of all degrees of freedom becomes valid. With respect to the convergence of this solution, moreover, remarks similar to those which were found at the conclusion of Section 18 on the perturbation of non-decomposed

systems are to be made since the series A_ϱ, B_ϱ are defined in the very same manner as series S is there.

In the manner described here the conclusions from Section 20 can also be completed if the random decomposition continues to exist in the perturbed motion. One should read 47 of Born's book on this subject[1]. It contains interesting quantum theoretical conclusions on the existence of phase relationships in Bohlian atoms.

24. The simultaneous existence of various types of decomposition. The simultaneous existence of various types of decomposition causes no new difficulties. They can be combined in many different manners. On the one hand, the limit-decomposed degrees of freedom can even be simultaneously decomposed characteristically or randomly. The first case only causes a simplification in the considerations from the previous figures, since H_o remains limited to the segment H_{oo}.[2] The second brings about no changes whatsoever since the characteristic peculiarity of the random decomposition $\partial H_0/\partial J_\varrho = 0$ in the previous section remains unimportant.

Furthermore, several degrees of freedom can be limit-decomposed simultaneously in a characteristic manner, while some can be limit-decomposed in a random manner. One must then create a completely developed initial movement by strict or approximated calculation of the secular perturbations. Since we must assume, nevertheless, that

1) M. Born, Atommechanik (cf. footnote in Sect. 1).

2) Cf. the detailed calculation by L. Nordheim, ZS. f. Phys. Vol. 17, p. 316, 1923, which is related to this case.

the limit decomposed degrees of freedom can be separated from the others, there is no difficulty in the combination of the methods.

25. The Delaunayian method[1]. Let us subsequently mention a short procedure which was a predecessor of the Bohlinian method and was also previously [2] transposed to quantum mechanics. It is the expedient by Delaunay for the illumination for perturbing commensurabilities, i.e., such members (9) in the series (8) of Section 18 which are characterised by particularly small denominators.

Let $$H \equiv H_0(J_k^0) + \lambda H_1(J_k^0, \tilde{w}_k^0) \qquad (1)$$

and let $w_\varrho^0 = \sum \tau_k w_k^0$ be a angle variable which is becoming secular, i.e., whose frequency ν_ϱ^0 in the unperturbed motion has a very small value which, thus, almost decomposes randomly. It is then possible to transpose the w_k^0 values into f - 1 variables w_α^0 and into w_ϱ^0 by a linear integral transformation with determinates ± 1. If one selects only those members of the perturbation function containing w_ϱ^0 --in our system of terminology the segment $\overline{H}_1$--the intermediary motion is with the Hamiltonian function

$$H_0^* \equiv H_0 + \lambda \overline{H}_1(\tilde{w}_\varrho^0) = W_0^* \qquad (2)$$

defined as that of a conditionally periodic system since only the variable pair w_ϱ^0, J_ϱ^0, aside from the constants J_α^0, is contained in H_0^* (cf. Section 5). The corresponding Hamiltonian differential

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1) Cf. Poincaré, Leçons Vol. I, conclusion.

2) By P. S. Epstein, ZS. f. Phys. Vol. 8, p. 211 and 305, 1922; Vol. 9, p. 22.

equation can thus be integrated and leads to new angle and action variables w_ϱ, J_ϱ, through the consideration of the rotation or libration of w_ϱ^0 in a familiar fashion. Thus, H contains the form

$$H \equiv H_0(J_\alpha^0, J_\varrho) + \lambda \tilde{H}_1,$$

in which the perturbation function no longer contains the troublesome members.

One can also use the procedure in order to take into account particularly predominant Fourier members of the perturbation function beforehand. In fact, one can relate member upon member of the same through repeated application to repeatedly new conditionally periodic intermediary motions. But, since by each new step infinitely many new Fourier members of the perturbation function are simultaneously introduced (in any event, they are of higher order in)[1], one can only integrate a small number of especially predominant perturbation members in this manner.

One sees that the Delaunayian method is actually only a special case of the Bohlinian procedure described in Section 20. Its peculiarity is that the differential equation for the secular perturbations can be strictly integrated through the extraction of a decomposed variable.

IV. PERTURBATION THROUGH CHRONOLOGICALLY VARYING FIELDS.

26. Non-closed systems. Section III was only concerned with

1) Cf. J. Woltjer, ZS. f. Phys. Vol. 31, p. 107, 1925.

closed systems whose Hamiltonian function was independent of time with the inclusion of the perturbation members and to which, thus, the form Chapter 3, Section 12, Equation (6) of the Hamiltonian partial differential equation could be applied. A series of problems--one need only recall the theory of dispersion and the action of thrusts on atoms as well as on the astronomical side the "limited" three-body problem[1]--make it desirable to turn away from this limitation. We thus observe the effect of small perturbations dependent on time of a system, which (aside from them) possesses a Hamiltonian function independent of time, and which proceeds through a conditionally periodic motion in the unperturbed case. Moreover, we limit ourselves to non-decomposed initial systems and differentiate both systems of multiple periodic or unperiodic perturbations.

Both cases can be visualized most easily when one assumes that the system S considered with the Hamiltonian function H_o (J^o) and with the angle and action variables w_k^o, J_k^o in the unperturbed case coupled with a second system $\mathfrak{S}$ which possesses the Hamiltonian function $\mathfrak{H}_o(q, p)$ without considering the interaction and also possesses, when it itself has conditionally periodic character,. the angle and action variables $\mathfrak{w}_k, \mathfrak{J}_k$. The coupling is expressed by interaction times of the common Hamiltonian function $\mathfrak{H}$ which can

1). Cf. Ch. 7, Sect. 26, 31f of this volume of the Handbook.

be order according to powers of a small parameter λ and can be written each time as periodic functions as w_k^0 :

$$\mathfrak{H} = H_0(J_0) + \mathfrak{H}_0(q, p) + \lambda H_1(J^0, w^0, q, p) + \cdots + \lambda^n H_n(J^0, w^0, q, p) + \cdots$$

The Hamiltonian function

$$H \equiv H_0(J^0) + \lambda H_1(J^0, w^0, q, p) + \cdots + \lambda^n H_n(J^0, w^0, q, p) + \cdots$$

which governs the system S is then an explicit function of time (if q_k, p_k can be dependent on time) as in the manner of $\mathfrak{H}_0$ of unperiodic or multiple periodic form. Thereby, q_k, p_k are absolutely fixed functions of t if the system $\mathfrak{S}$ is not noticeably influenced by the coupling, e.g., when it is so much greater that S that its counter action can be ignored.

27. Multiple periodic dependence on time of the perturbation function. Let the perturbation function be given by the Hamiltonian function

$$H \equiv H_0(J^0) + \lambda H_1(J^0, w^0, w) + \cdots + \lambda^n H_n(J^0, w^0, w) + \cdots,$$

wherein

$$w_k = n_k t + b_k$$

(we do not explicitly introduce the constant γ_k values), and

$$H_n = \sum \cdots \sum A^{(n)}_{r_1 \ldots r_f t_1 \ldots t_l} e^{2\pi i (r_1 w_1^0 + \ldots + r_f w_f^0 + t_1 w_1 + \ldots t_l w_l)}.$$

The canonical equations are

$$\dot{w}_k^0 = \frac{\partial H}{\partial J_k^0}, \qquad \dot{J}_k^0 = -\frac{\partial H}{\partial w_k^0}.$$

If one assumes that the perturbed total system is conditionally periodic and that the new angle variables are thus bound to equations of form (cf. Sect. 14 (3))

$$w_k = w_k^0 + \lambda(\tilde{w}_1^0, \dots \tilde{w}_f^0, \tilde{w}_1, \dots \tilde{w}_f)$$

then for the generatrix S of the canonical transformation $(w^o, J^o) \rightarrow (w, J)$

$$S = \sum J_k w_k^0 + \lambda S_1(\tilde{w}^0, \tilde{w}) + \cdots + \lambda^n S_n(\tilde{w}^0, \tilde{w}) + \cdots,$$

and

$$J_k^0 = J_k + \lambda \frac{\partial S_1}{\partial w_k^0} + \cdots$$

appears. If one expands the new Hamiltonian function $H + \partial S/\partial t$ at point J_k by $J_k^o - J_k$, then

$$\begin{aligned} H + \frac{\partial S}{\partial t} \equiv H_0(J) &+ \lambda\left\{\sum_1^f \nu_k^0 \frac{\partial S_1}{\partial w_k^0} + \sum_1^f n_k \frac{\partial S_1}{\partial w_k} + H_1(J, w^0, w)\right\} \\ &+ \cdots + \lambda^n\left\{\sum_1^f \nu_k^0 \frac{\partial S_n}{\partial w_k^0} + \sum_1^f n_k \frac{\partial S_n}{\partial w_k} + \Phi_n(J, w^0, w)\right\} + \cdots \\ &= W_0 + \lambda W_1(t) + \cdots + \lambda^n W_n(t) + \cdots \end{aligned}$$

This is a differential equation of the type in Chapter 3, Section 12, Equation (8); thereby, $H + \partial S/\partial t$ is not arbitrarily set = 0, but is equal to a constant W_o plus a power series in λ of once arbitrary functions of t (or of the n_k values). W_o results from the requirement that S_o be independent of t and that the w_k, J_k values should agree for λ = 0 with the w_k^o, J_k^o values. The members of the power series can be made constant through the proper selection of $(\partial S_j/\partial w_k)$. For this, one only needs to meet the requirement that, for example

$$\sum_1^f \nu_k^0 \frac{\partial S_1}{\partial w_k^0} + \sum_1^f n_k \frac{\partial S_1}{\partial w_k} + \tilde{H}_1 = 0;$$

then $H_1 = W_1$. remains. (The bar indicates, as always, averaging across the unit q of the w_k^o values.) ηW_1 is left the chronological mean value of the interaction energy of the first order across the unperturbed path.

A description of a further calculation will not be given. It is simply a logical transposition of the procedure from Section 18 of the stationary perturbation of a non-decomposed system.

Such calculations have been repeatedly described in connection with the dispersion series[1]. System S is then conceived of as a union of point-like electric loads which are dynamically bound in some way or other, e.g. in the manner of the Bohrian atom, under the influence of a quickly changing outer field which, however, is homogeneous in the first approximation. The dipole moment of the system under consideration $\mathfrak{p}^o$ is represented in the unperturbed case as a multiple Fourier series of its old angle variables

$$\mathfrak{p}^0 = \sum \cdots \sum a_{\tau_1 \ldots \tau_f} e^{2\pi i (\tau_1 w_1^0 + \cdots + \tau_f w_f^0)},$$

the light wave as a multiple periodic function of time

$$\mathfrak{E} = \sum \cdots \sum e_{t_1 \ldots t_f} e^{2\pi i (t_1 w_1 + \cdots + t_f w_f)}.$$

H_1 becomes the scalar product $\mathfrak{p}^0 \mathfrak{E}$. If the calculation scheme described above is applied in such a form to perturbations which cannot be considered as couplings with a system $\mathfrak{S}$ at a conditionally periodic total mechanism, then there exists a compulsion expressed in the

1) Cf., e.g., the studies cited in Sect. 25, footnote 2, and in Sect. 28, which contain further suggestions.

in the previous section if the angle and action variables w_k^o, J_k^o of the unperturbed system are introduced so that

$$H \equiv H_0(J^0) + \lambda H_1(J^0, w^0, t) + \cdots + \lambda^n H_n(J^0, w^0, t) + \cdots \tag{1}$$

The Hamiltonian partial differential equation is of the variety in Chap. 3, Sect. 12, Equation (8), and leads to an action function

$$S(\alpha, w^0, t) \equiv S_0 + \lambda S_1 + \cdots + \lambda^n S_n + \cdots$$

with f integration constants α_k, of which we can assume that they are transformed into the quantity J_k^o for $\lambda = 0$

$$J_k^0 = \alpha_k + \lambda \frac{\partial S_1}{\partial w_k^0} + \cdots \tag{3}$$

Thus,
$$S_0 = \sum_k \alpha_k w_k^0$$

and for the variables β_k conjugated conically to the α_k values, the derivation

$$\beta_k = w_k^0 + \lambda \frac{\partial S_1}{\partial \alpha_k} + \cdots$$

Naturally, one cannot assume that the perturbed system is still multiple periodic; the new variables were thus not indicated with w_k, J_k.

If one introduces in the usual way and with the help of (3) the constant α_k into (1) and expands again by λ, then

$$H + \frac{\partial S}{\partial t} \equiv H_0(\alpha) + \sum_{n=1}^{\infty} \lambda^n \left\{ \sum_k \nu_k^0 \frac{\partial S_n}{\partial w_k^0} + \frac{\partial S_n}{\partial t} + \Phi_n(\alpha w^0 t) \right\},$$

whereby the function Φ_n are uncharacteristic Fourier series of type (2)

$$\Phi_n = -\sum_{-\infty}^{+\infty} \cdots \sum_{-\infty}^{+\infty} C^{(n)}_{\tau_1 \cdots \tau_f}(t)\, e^{2\pi i (\tau_1 w_1^0 + \cdots + \tau_f w_f^0)}.$$

One again sees that use is made of the freedom mentioned in Chap. 3, Sect. 12 with respect to the dependency on time of $H + \partial S/\partial t$ through the selection of S_o, insofar as

$$H + \frac{\partial S}{\partial t} = W_0 + \lambda W_1(t) + \cdots$$

with

$$W_0 = H_0(\alpha) \neq 0$$

The other functions $W_n(t)$ must be caused to vanish, however, by the suitable selection of the S_n values. This requires

$$\sum_k \nu_k^0 \frac{\partial S_n}{\partial w_k^0} + \frac{\partial S}{\partial t} = -\Phi_n = \sum \cdots \sum C^{(n)}_{\tau_1 \cdots \tau_f}(t)\, e^{2\pi i (\tau_1 w_1^0 + \cdots + \tau_f w_f^0)},$$

an equation which, as Jordan as shown, is fulfilled by the postulate

$$S_n = \sum \cdots \sum e^{2\pi i (\tau_1 w_1^0 + \cdots + \tau_f w_f^0)} \int_{t_0}^{t} e^{-2\pi i (\tau_1 \nu_1^0 + \cdots + \tau_f \nu_f^0)(t - t')} C^{(n)}_{\tau_1 \cdots \tau_f}(t')\, dt'.$$